Balanced weights on trees and boundary classes of stable curves

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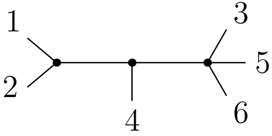
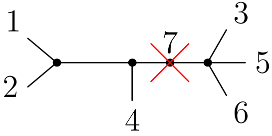
On joint work with Jake Levinson (University of Montreal)

MSU Combinatorics and Graph Theory Seminar, October 11, 2023

Combinatorial warmup: Stable trees

- A **stable tree**, or **at-least-trivalent tree**, is a leaf-labeled tree with leaves $1, 2, \ldots, n$ and no vertices of degree 2. Ex and non-ex:





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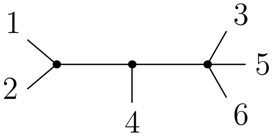
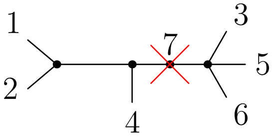
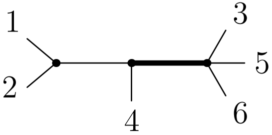
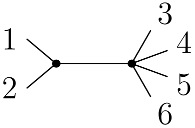




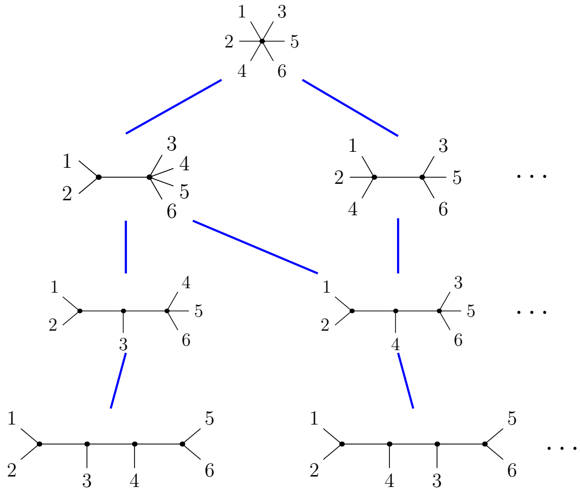
- **Contraction** of a non-leaf edge still results in a stable tree:





Poset ST_n of stable trees

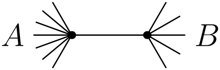
Poset ST_n: tree $T_1 > T_2$ if T_2 contracts to T_1. Portion of ST_6:



ST_n is a join-semilattice, coatomic. $(2n-5)!!$ minima (trivalent)

When do meets exist?

- **Coatoms** of ST_n look like:



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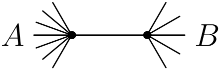
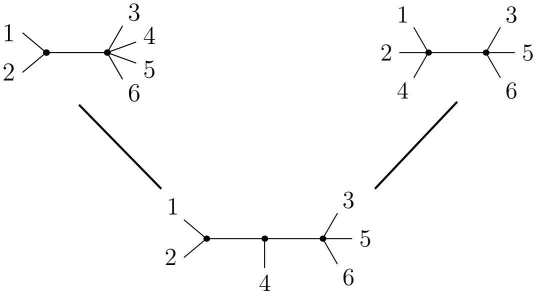


  Write as $D(A|B)$.
- **Compatibility:** $D(A|B) \wedge D(X|Y)$ exists iff either $A \subseteq X$, $A \subseteq Y$, $A \supseteq X$, or $A \supseteq Y$ (equivalent up to symmetry).
- **Example:**



Coatomicity

- Let T be a stable tree, e internal edge of T, let A_e, B_e be the sets of leaves on either side of e in T. Write $D_e = D(A_e|B_e)$.

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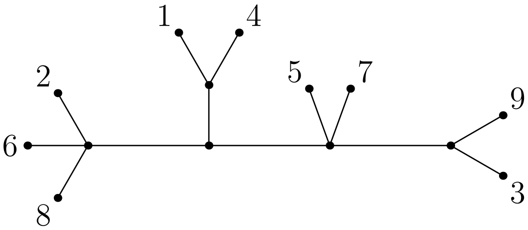

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$$D(268|134579) \wedge D(14|2356789) \wedge D(12468|3579) \wedge D(1245678|39)$$

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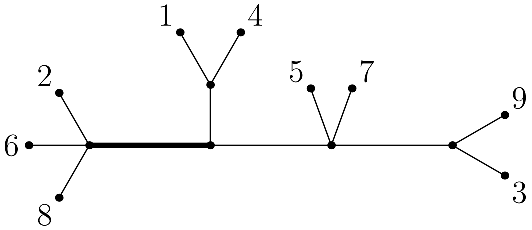

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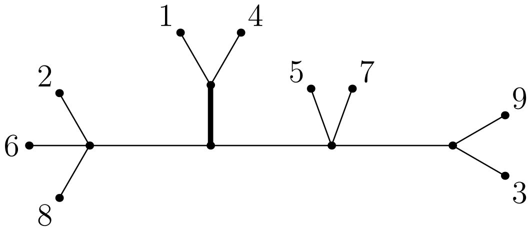

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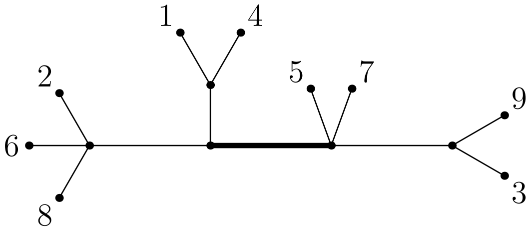

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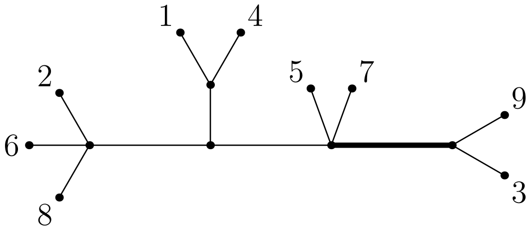

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General meets

Proposition ("G., Levinson", "folklore (Giansiracusa)")

The meet $T \wedge T'$ exists iff for all internal edges $e \in T$ and $e' \in T'$, $D_e \wedge D_{e'}$ exists.

Alternatively: simplicial complex whose vertices are coatoms and faces are meets is a **flag complex**: a face is in the complex if and only if all its edges are.

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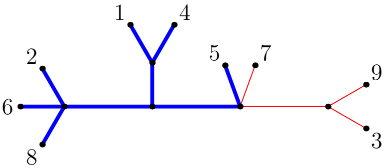
Corollary

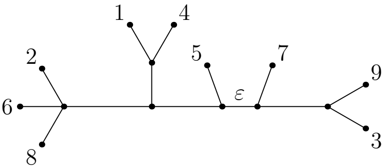
A collection of trees $T_1, \ldots, T_n$ has a meet if and only if each pairwise meet $T_i \wedge T_j$ exists.

Proof sketch/construction of meets

- **Lemma:** $T \wedge D(A|B)$ exists iff $D(A|B)$ compatible with every divisor D_e of an internal edge e in T.
- **Construction:** Find the vertex v such that the branches at v either only contain A leaves or only B leaves; form $S = T \wedge D(A|B)$ by inserting a new edge at v separating A, B (or $S = T$ if result is not stable).

Example. T at left below; $T \wedge D(124568|379)$ shown at right.





Geometric motivation: Rational curves

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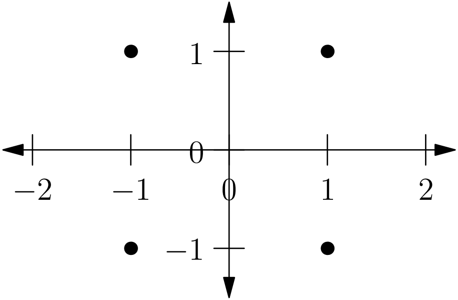

- Work over $\mathbb{C}$, curves are genus 0, define $\overline{M}_{0,4}$ to be the **moduli space** of these curves. $\overline{M}_{0,4} \cong \mathbb{P}^1$; above curves are $(1-t)x^2 + ty^2 = 1$

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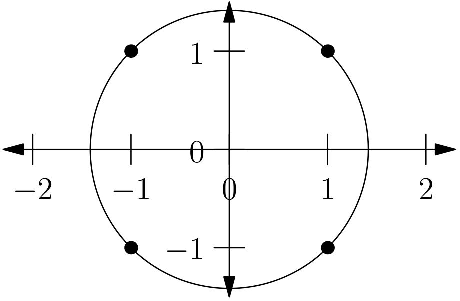

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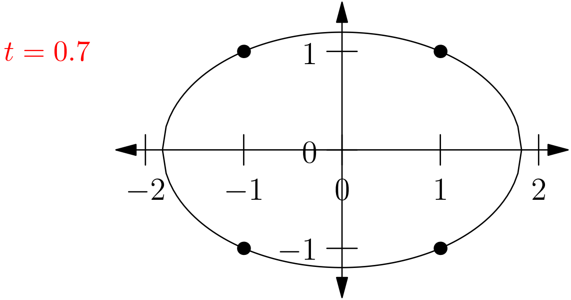

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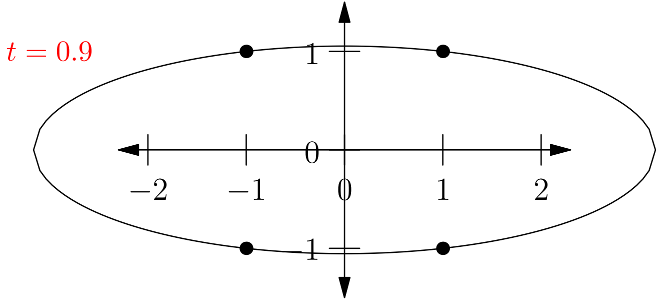

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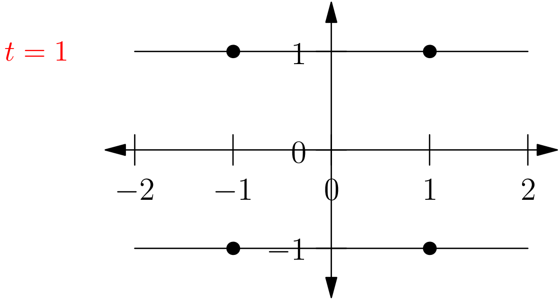

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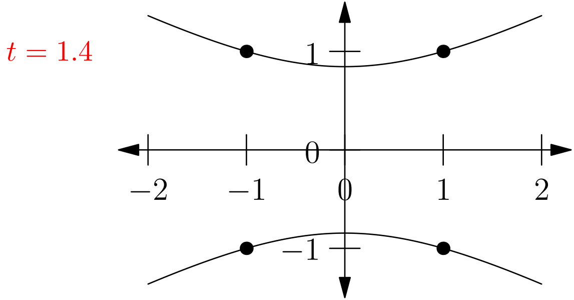

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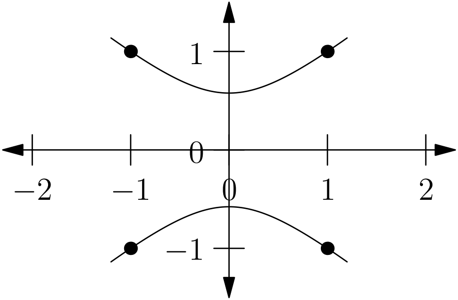

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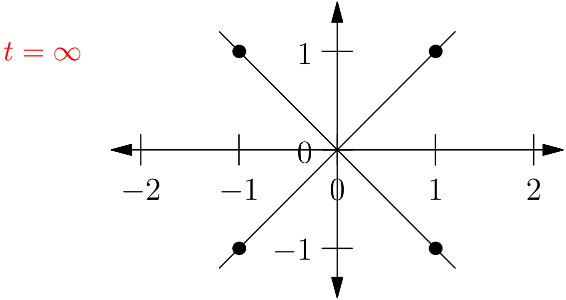

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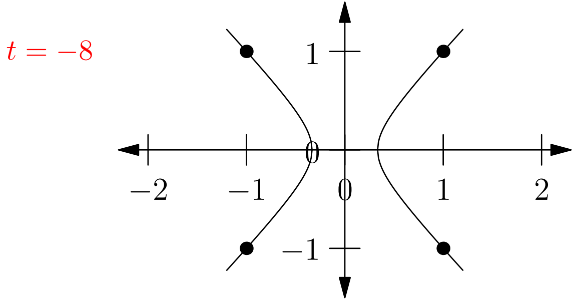

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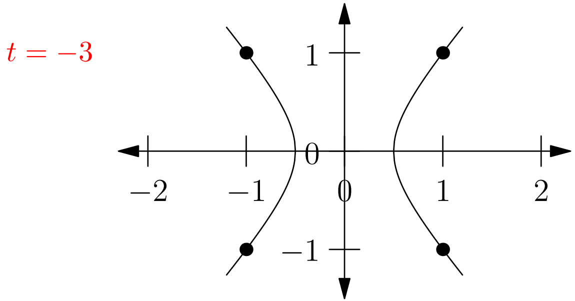

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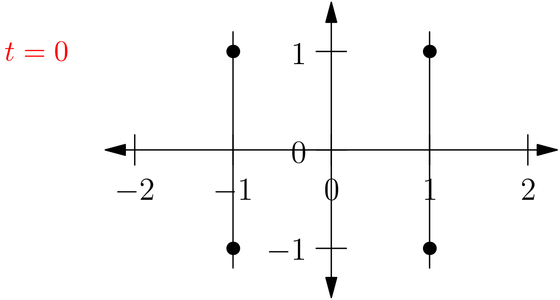

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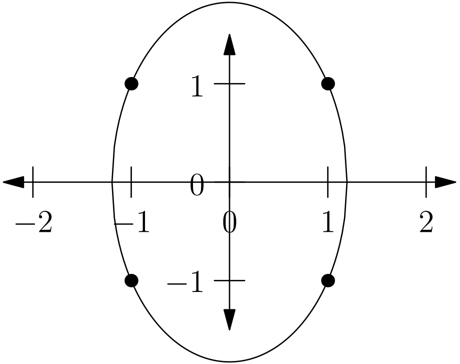

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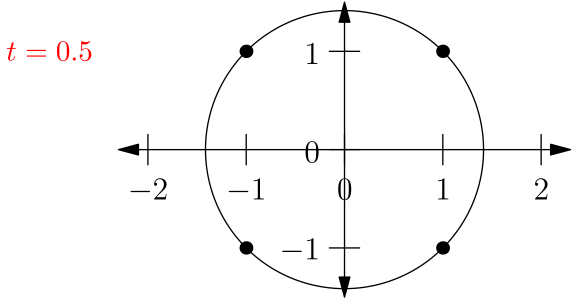

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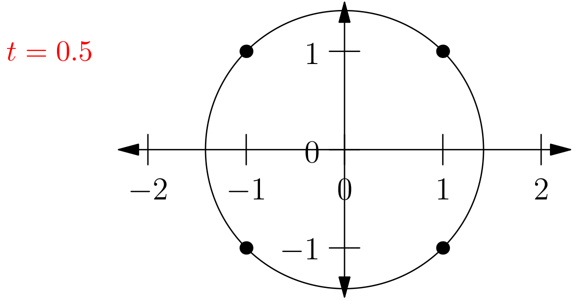

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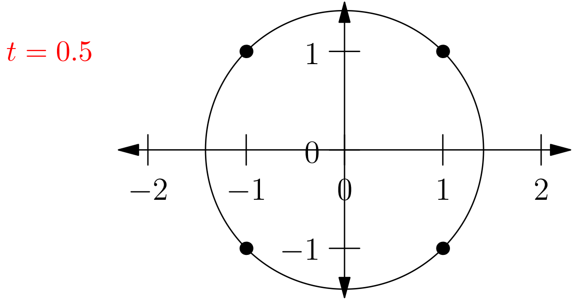

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- Quartics through 6 general points in $\mathbb{P}^4$? $\overline{M}_{0,6}$

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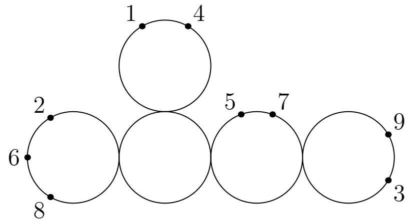
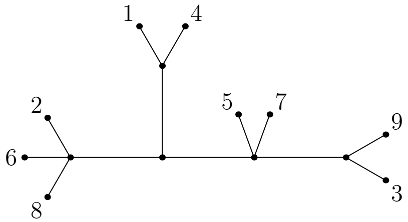
- Alt def: $\overline{M}_{0,n}$ is space of **stable curves** of genus 0, n marked points





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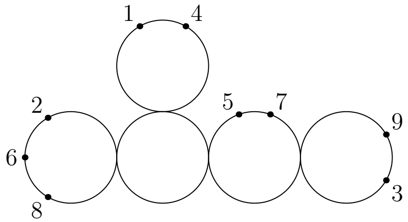

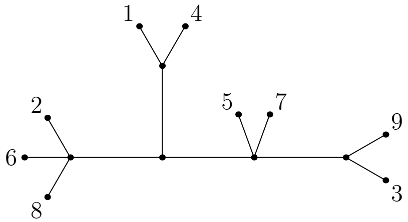

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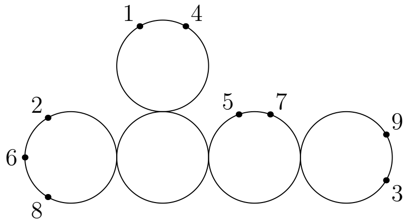
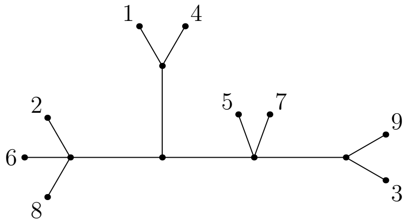
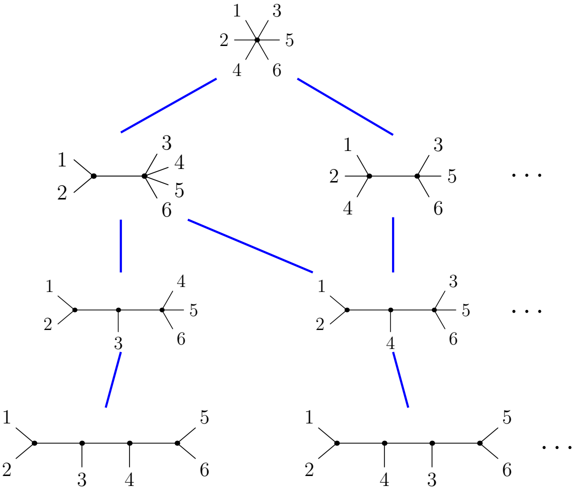
- **Stable curve:** Tree structure of $\mathbb{P}^1$'s glued at nodes, each has at least 3 special points
- **Dual tree:** Vertex for each $\mathbb{P}^1$, edges are incidence
- **Boundary stratum:** X_T is closure of set of curves with dual tree T
- $X_T \subseteq X_{T'}$ iff $T < T'$ in ST_n.





Poset ST_n again

Poset ST_n is poset of strata X_T under inclusion:



Chow ring and intersection theory on $\overline{M}_{0,n}$

- **Chow ring:** Graded ring $A^*(\overline{M}_{0,n})$, elements are rational equivalence classes $[X]$ of closed subvarieties X (like boundary strata)
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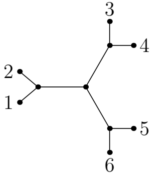
- In dim 0: $[D(12|3456)] \cdot [D(1234|56)] \cdot [D(34|1256)] = [T] = [pt]$,

$T =$ 

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Chow ring and intersection theory on $\overline{M}_{0,n}$

- **Chow ring:** Graded ring $A^*(\overline{M}_{0,n})$, elements are rational equivalence classes $[X]$ of closed subvarieties X (like boundary strata)
- Product gives intersections; if $D(A|B) \neq D(X|Y)$ then $[D(A|B)] \cdot [D(X|Y)]$ is $[X_T]$ where $T = D(A|B) \wedge D(X|Y)$
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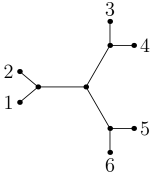
$T =$ 

- When a product lands in dimension 0, it is $c \cdot [pt]$ for some c, called the **intersection number**.
- What are intersection numbers of arbitrary products of $[X_T]$'s?

Self-intersections, Psi classes, and tangency

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- **Formula:** Interpreting $\psi_\bullet$ on $\overline{M}_{0,A\cup\bullet}$ or $\overline{M}_{0,B\cup\bullet}$ as pushed forward to a class on $\overline{M}_{0,n}$, have

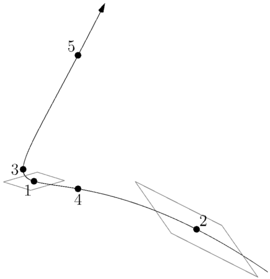
$$[D]^2 = -(\psi_{A,\bullet} + \psi_{B,\bullet})$$

In fact

$$[D]^{k+1} = (-1)^k(\psi_{A,\bullet} + \psi_{B,\bullet})^k$$

Psi class intersection numbers

- **Rational curve interpretation:** ψ_i is class of degree $n-2$ curves through n points in $\mathbb{P}^{n-2}$ tangent to a given hyperplane at point i
- **Example:** $\psi_1\psi_2$ in $\overline{M}_{0,5}$ is class of cubics through 5 points in $\mathbb{P}^3$ tangent to fixed hyperplanes at $1, 2$. Two solutions; $\psi_1\psi_2 = 2 \cdot [pt]$.



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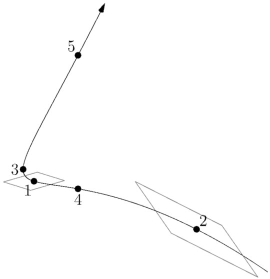

- **General intersection number:** if $a_1 + \cdots + a_n = n-3$, have

$$\psi_1^{a_1} \cdots \psi_n^{a_n} = \binom{n-3}{a_1, a_2, \ldots, a_n} \cdot [pt]$$

Intersections of boundary classes

- **Setup:** Stable trees $T_1, \ldots, T_\ell$ with $\sum \mathrm{codim}(X_{T_i}) = \dim \overline{M}_{0,n}$ (i.e., total $n-3$ internal edges) and $\bigcap X_{T_i} = X_T$.
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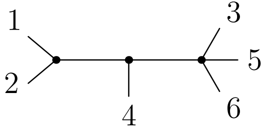

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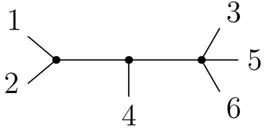
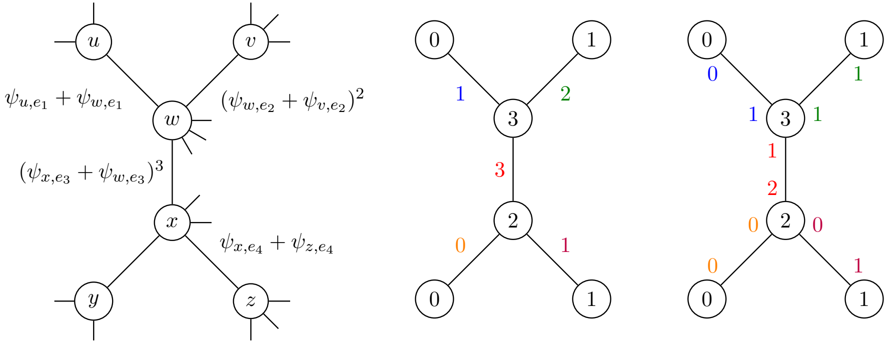


- Write $\psi_{v,e}$ for the psi class at edge e from component v. Then

$$\prod [X_{T_i}] = \prod_{e \in T} [D_e]^{k(e)+1} = \prod_{e=(v,w)\in T} (-1)^{k(e)} (\psi_{v,e} + \psi_{w,e})^{k(e)}$$

Expansion and balanced weights

- **Claim**: Only one term in expanding $\prod(-\psi_{v,e} - \psi_{w,e})^{k(e)}$ survives!



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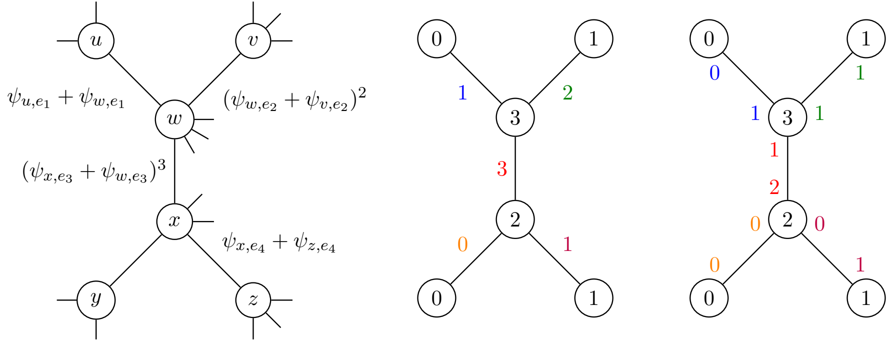

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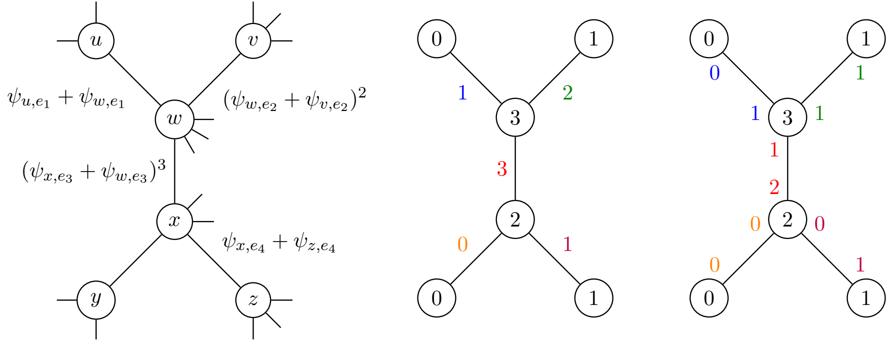

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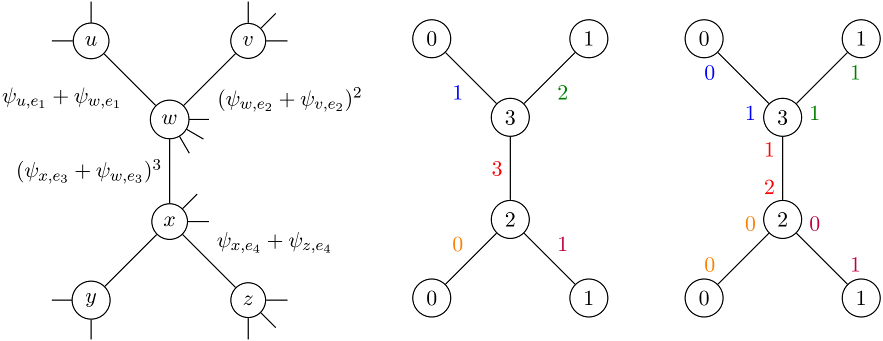
- Vertex sums are $\deg(v) - 3$; unique splitting by greedy algorithm



Balanced Weights formula

- **Summary:** Left with a unique term in expansion:

$$(-1)^{\sum k(e)} \prod_e \binom{k(e)}{k(v,e),k(w,e)} \prod_{v,e} \psi_{v,e}^{k(v,e)}$$

- **Product of psi classes**: multinomial coefficient for each vertex v:

$$\prod_i [X_{T_i}] = (-1)^{\sum k(e)} \prod_{e\in T} \binom{k(e)}{k(v,e)} \prod_{v\in T} \binom{\deg(v)-3}{k(v,e_1),k(v,e_2),\dots}$$

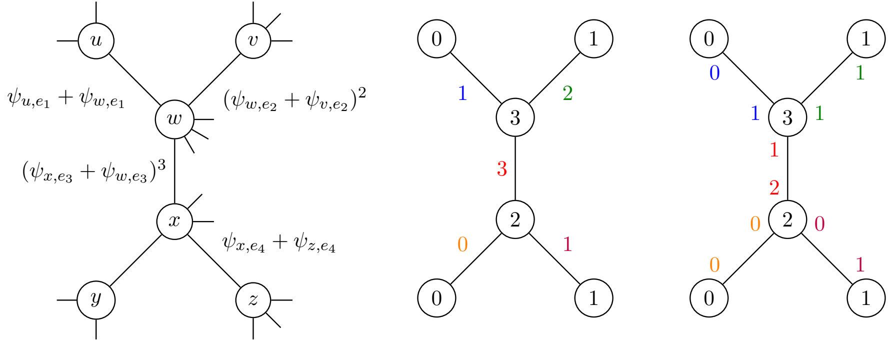
Simplified: Let $n(v) = \deg(v) - 3$ for any v. Then:

Theorem (G., Levinson, "folklore")

We have $\prod_i [X_{T_i}] = (-1)^{\sum n(v)} \frac{\prod n(v)! \prod k(e)!}{\prod k(v,e)!^2} \cdot [pt]$

Example of Balanced Weights Formula

Formula: $\prod_i [X_{T_i}] = (-1)^{\sum n(v)} \frac{\prod n(v)! \prod k(e)!}{\prod k(v,e)!^2} \cdot [pt]$



$$(-1)^7 \frac{(3! \cdot 2! \cdot 1! \cdot 1! \cdot 0! \cdot 0!)(3! \cdot 2! \cdot 1! \cdot 1! \cdot 0!)}{(2! \cdot 1! \cdot 1! \cdot 1! \cdot 1! \cdot 0! \cdot 0! \cdot 0! \cdot 0!)^2} = \frac{-6 \cdot 2 \cdot 6 \cdot 2}{2^2} = -36.$$

Examples in special cases

- **Divisor power:** (binomial coefficient)

$$[D(1\,2\,3\,4\,5\,6\,7|8\,9\,10\,11\,12)]^8 = \binom{8}{5}[pt] = 56[pt]$$

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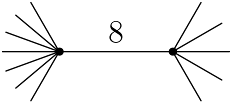
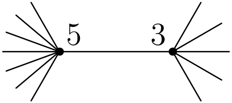
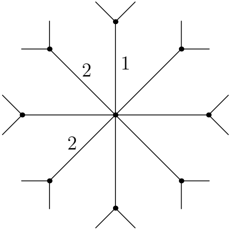




- **Star crab:** (multinomial coefficient)



$$-\binom{5}{2,2,1}[pt] = -30[pt]$$

Announcement: COMOC 2024

COMOC: Combinatorics Of Moduli Of Curves - workshop at Banff International Research Station (BIRS), 2024

Dates: July 28-Aug 2, 2024

Purpose: Bring together geometers and combinatorialists to work on combinatorial problems on moduli spaces of curves

Applications: We have a small number of spots open for applications - spread the word!

Website: https://sites.google.com/view/comoc2024/home

Thank you!