Twisting (and braiding?) open positroid varieties

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Grassmannian $Gr(k,n) := \{ W \subset \mathbb{C}^n : \dim W = k \}$

pick a basis $\left\{ = GL_k \backslash \mathrm{Mat}^\circ(k,n) \ni M \right.$ (Mat° means rank equals k)

Plücker coords : Δ_I = $k \times k$ minor of M in cols I

$\binom{n}{k}$ #'s which characterize your k-subspace; satisfy famous "Plücker relns"

e.g. $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$.

Matroid of $x \in Gr(k,n)$ is $\{ I \in \binom{[n]}{k} : \Delta_I(x) \neq 0 \}$. Grouping x's according to their

matroid yields $Gr(k,n) = \coprod_{\text{matroids } \mathcal{M}} \underbrace{\{ x : \text{matroid}(x) = \mathcal{M} \}}_{\text{matroid stratum}}$

very poorly behaved ! — bad singularities (Mnev's thm)
— strata are reducible + non-equidimensional

$Gr(k,n)_{\geq 0}$: subset of $Gr(k,n)$ on which all $\Delta_I \in \mathbb{R}_{\geq 0}$.

Positroid : a matroid of the form matroid(x) : $x \in Gr(k,n)_{\geq 0}$ } positroids $\subsetneq$ realizable matroids $\subsetneq$ matroids

Lusztig, Postnikov

The decomposition $Gr(k,n)_{\geq 0} = \coprod_{\substack{\text{positroids}\\ \mathcal{M}}} \{x \in Gr(k,n)_{\geq 0} : \text{matroid}(x) = \mathcal{M}\}$ <u>is</u> well-behaved.

- each piece is topologically an open ball "positroid cell", ∂ of each cell is union of lower-dim'l cells
- <u>regular</u> CW complex
- $Gr(k,n)_{\geq 0}$ itself is a closed ball — Galashin Karp Lam
- we know how to index the cells (i.e. the positroids), dim of each cell, cell closure order, ...

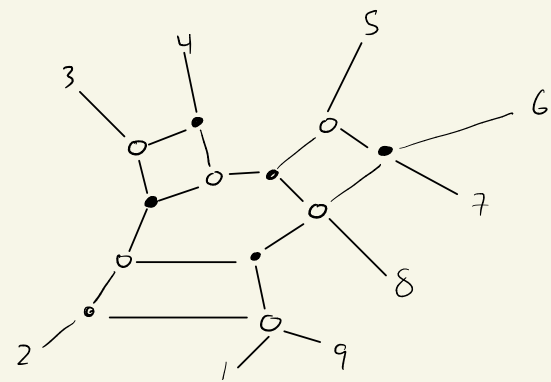
Objects that index positroids:

1) Grassmann necklaces $\mathcal{I} = (1236, 2346, 3456, 4567, 5679, 6789, 3789, 3689, 2369)$

2) Decorated permutations $\pi : 4\,5\,7\,9\,8\,3\,6\,2\,1$ (decorate: 2-color fixed points)

3) Reduced plabic graphs modulo move-equivalence

trip permutations

$G =$ 

left at $\circ$

right at $\bullet$

$\exists$ bijections

$\mathcal{M} \longleftrightarrow \mathcal{I} \longleftrightarrow \pi \longleftrightarrow G/_{\text{moves}}$

"Classical" Question: How many minors must be positive to guarantee that all $\Delta_I(x) \in \mathbb{R}_{>0}$?

Answer: Only $k(n-k)+1$ many (not $\binom{n}{k}$ many!)

Example $(k=4, n=8)$ $k(n-k)+1 = 17$ vs $\binom{8}{4} = 70$

Idea of proof: Plü coords satisfy alg. identities, using these identities can write certain Plü coords as subtraction-free expressions in others, e.g.

$$\Delta_{24} = \frac{\Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}}{\Delta_{13}}, \quad \Delta_{25} = \frac{\Delta_{12}\Delta_{15}\Delta_{34} + \Delta_{14}\Delta_{15}\Delta_{23} + \Delta_{12}\Delta_{13}\Delta_{45}}{\Delta_{13}\Delta_{14}}$$

$\exists$ "magical" collections $\mathcal{C}$ of $k(n-k)+1$ many Plü coords w/ prop. that every other Δ_I is a subtracn free expression in the elts of $\mathcal{C}$.

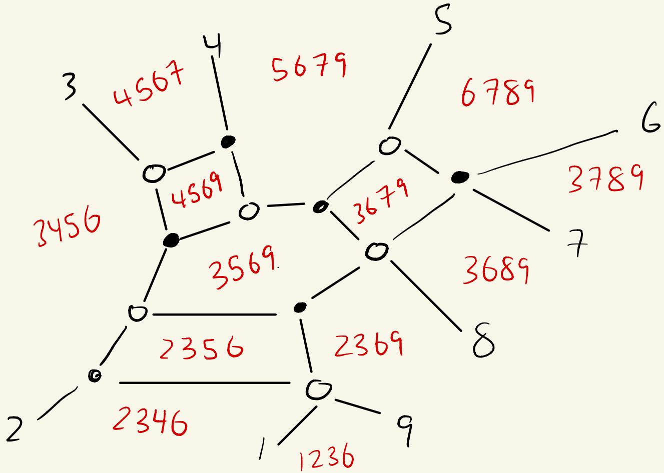
know which minors $=0$ vs $\neq 0$

Variation on this question: Fix positroid $\mathcal{M}$ and $x \in Gr(k,n)$ with $\text{matroid}(x) = \mathcal{M}$.

How many ... all $\Delta_I(x) \in \mathbb{R}_{\geq 0}$?

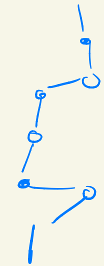
Answer: $\dim(\text{cell}(\mathcal{M}))+1$ many.

Construction of efficient TP tests:



"face collection ~~of~~ G"

put 4's in faces left of



$\underline{\text{Thm}}$ (Postnikov, Galashin Lam ..) If $\underbrace{G}_{\text{plabic graph}} \longleftrightarrow \underbrace{\mathcal{M}}_{\text{positroid}}$ then its face collection is a TP test for cell $(\mathcal{M})$.

$\underline{\text{Def'n}}$ $I, J \in \binom{[n]}{k}$ are weakly separated if $I \setminus J$ and $J \setminus I$ are cyclically separated.

e.g. 13458 and 12459

Easy: Any such face collection $\mathcal{C}$ is pairwise weakly separated and satisfies

$$\mathcal{I} \subseteq \mathcal{C} \subseteq \mathcal{M} \quad \text{where} \quad \underbrace{\mathcal{I}}_{\text{Gr. necklace}} \longleftrightarrow \underbrace{\mathcal{M}}_{\text{positroid}}$$

Thm (Oh-Postnikov-Speyer) Every such $\mathcal{C}$ of $\underbrace{\text{max'l size}}_{\dim(\mathrm{cell}(\mathcal{M}))+1}$ is the face collection of some G for $\mathcal{M}$.

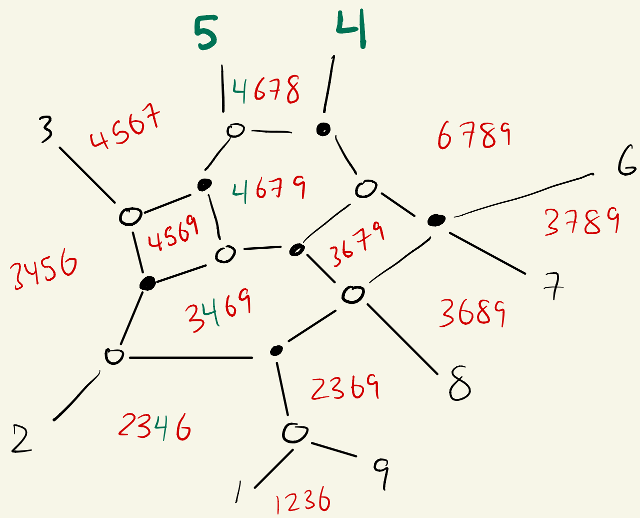
<u>Problems</u> for many $\mathcal{M}$, $\exists!$ plabic graph G for this positroid. ($\Rightarrow$ "few" TP tests)

The condition $\mathcal{C} \subseteq \mathcal{M}$ is needed (vars in $\mathcal{C}$ should be positive, i.e. in $\mathcal{M}$).

The condition $\mathcal{I} \leq \mathcal{C}$ is <u>too restrictive</u>: in examples, one finds other Plücker TP tests which don't satisfy this

We relax this condition + find many more Plü TP tests.

Def'n For a plabic graph G with n ∂ vertices and for $\rho \in S_n$, have **relabeled plabic graph** G^ρ in which ∂ vertices are relabeled according to ρ.



A graph relabeled by $\rho = s_4$.

1) Trip permutation still makes sense:

$$\underbrace{\mathrm{TripPerm}(G^\rho)}_{\pi} = \rho\, \underbrace{\mathrm{TripPerm}(G)}_{\mu}\, \rho^{-1}$$

2) ∂ faces will be a Grassmann "like" necklace in which you remove $\rho(i)$, not i, at i^{th} step

3) $\mathrm{FaceCollection}(G^\rho) = \rho\,(\mathrm{FaceCollection}(G))$

Metathm In easily checkable situations, G^ρ determines a TP test for cell $(\mathcal{M})$, where $\underbrace{\mathcal{M}}_{\text{positroid}} \longleftrightarrow \underbrace{\mathrm{TripPerm}(G)}_{\text{permutation}}$

Positroid variety $\Pi_M = \{ x \in Gr(k,n) : \Delta_I(x) = 0 \ \forall I \notin M \}$

$= \text{Zariski-closure of } \mathrm{cell}(M).$

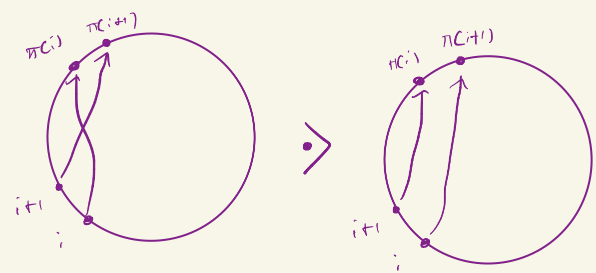
Open positroid var $\Pi_M^\circ = \{ x \in \Pi_M : \Delta_I(x) \neq 0 \ \forall I \in \mathcal{I} \}$ $\quad (\underbrace{\mathcal{I}}_{\text{necklace}} \longleftrightarrow \underbrace{M}_{\text{positroid}})$

Have $Gr(k,n) = \bigcup_M \Pi_M = \bigsqcup_M \Pi_M^\circ$ and this decomposition is nice — Π_M° is smooth, irreducible, ...

Poisson geom., Frob. splittings, (q,t)-Catalan #'s, ...

If $\underbrace{G}_{\text{graph}} \longleftrightarrow \underbrace{M}_{\text{positroid}}$ then any choice of nonzero cx #'s for Faces(G) determines a point in Π_M°, moreover this recipe is injective. (part of def'n of "cluster")

Def'n partial order $\leq$ on trip permutations whose cover relns are



"weak circular order, weak Bruhat order"

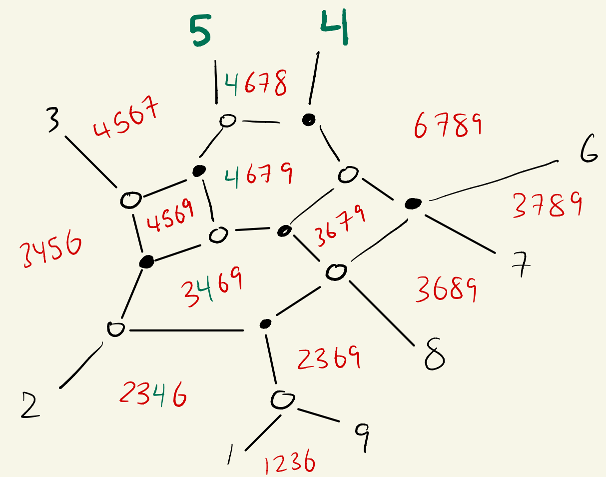
Thm(F., SB) Assume 0) $\pi_P \leq \pi$. (this guarantees that ∂ faces of G^P are non-vanishing on Π°_μ where $\mu \leftrightarrow \pi$).

Then TFAE: 1) Faces(G^P) are a cluster in a cluster structure on the open positroid var. Π°_π (in a particular, a TP test).

2) ∂ Faces of G^P are pairwise weakly sep'd. 2') all faces are...

G^P has the "right #" of faces $\{$ 3) $\dim \Pi^\circ_\pi = \dim \Pi^\circ_\mu$ where $\pi =$ TripPerm(G^P) $\mu =$ TripPerm(G)

4) $\Pi^\circ_\pi \simeq \Pi^\circ_\mu$.



Rmk 1) The cluster structure on Π°_μ coming from a relabeled plabic graph is not the same as (mutation-equivalent to) the cluster structure from ordinary plabic graphs.

We conj. that it is a certain "rescaling" of this structure.

cf. twist automs of BFZ, Marsh Scott, Muller Speyer

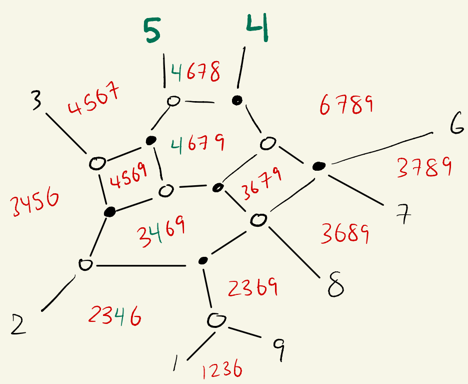
2) We show $\tilde{\Pi}^\circ_\pi \xrightarrow{\sim} \hat{\Pi}^\circ_\mu$ via the following "twist isomorphism":

Let $x \in \tilde{\Pi}^\circ_\pi$ be given by $k \times n$ matrix $[v_1 \cdots v_n]$.

Let $I_i = \{p(i), i_2, \ldots, i_k\}$ be i^{th} ∂ face of G^P.

$$v_i \longmapsto v_{i_2} \wedge \cdots \wedge v_{i_k} \in \Lambda^{k-1}\mathbb{C}^k \simeq \mathbb{C}^k.$$

Resulting matrix lies in $\hat{\Pi}^\circ_\mu$ and this is an isomorphism.



$$\underset{\tilde{\Pi}^\circ_\pi}{\underset{\cap}{[v_1, \ldots, v_9]}} \longmapsto \underset{\hat{\Pi}^\circ_\mu}{\underset{\cap}{[v_2\wedge v_3\wedge v_6,\ v_3\wedge v_4\wedge v_6,\ v_4\wedge v_5\wedge v_6,\ v_4\wedge v_6\wedge v_7, \ldots]}}$$

Rmk In example above, the defining conditions of Π_π are that

$v_4 \in \mathrm{span}(v_1, v_2, v_3)$ $\quad v_5 \in \mathrm{sp}(v_2, v_3, v_4)$ $\quad v_8 \in \mathrm{sp}(v_5, v_6, v_7)$ $\quad v_2 \in \mathrm{sp}(v_8, v_9, v_1)$ $\quad v_9 \parallel v_1$

Those for Π_μ are

$v_4 \in \mathrm{span}(v_2, v_3)$ $\quad v_5 \in \mathrm{sp}(v_1, v_2, v_3)$ $\quad v_8 \in \mathrm{sp}(v_5, v_6, v_7)$ $\quad v_2 \in \mathrm{sp}(v_8, v_9, v_1)$ $\quad v_9 \parallel v_1$

These conditions don't match up nicely, but the above isom. accomplishes this.

Bonus topic: automorphisms of $\Pi^\circ_{\mathcal{M}}$ from braids.

For $I \in \binom{[n]}{k}$ with $i \in I \not\ni i-1$ define column-switching map

$$\tau_i : \left[v_1 \; v_2 \; \cdots \; v_n \right] \longmapsto \left[v_1 \cdots , \; v_i, \; \frac{\Delta_{I \setminus i \cup i-1}}{\Delta_I} v_i - v_{i-1}, \; \cdots \right]$$

We'll apply this construction when $I = I_i$ is i^{th} term of Gr necklace $\mathcal{I}$.

(so implicitly, τ_i depends on $\mathcal{I}$!)

Thm (F., Keller) Let S be a subset such that the transpositions $(i-1\; i) : i \in S_n$ pairwise commute.

Suppose $\prod_{i \in S} (i-1\; i)$ commutes with π.

Then $\prod_{i \in S} \tau_i$ is an automorphism of $\Pi^\circ_{\mathcal{M}}$ where $\mathcal{M} \leftrightarrow \mathcal{I}$.

"compatible with cluster structures"

e.g. $\pi = 457983621$ and $S = \{2, 5, 9\}$

Rk 1) The τ_i's satisfy braid relns, so we get a subgroup of the braid group acting by automorphisms.

2) This generalizes an older thm of mine from the top-dimensional pos. var.

$\pi = k+1, \ldots, n, 1, \ldots k$

to all pos. vars.

3) This thm can be used to easily construct ∞ many clusters / TP tests,

∞ many Lagrangian fillings of Legendrian knots, ... Casals-Gao

Thanks!