Parabolic Tamari lattice of type B, and more

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arXiv:2112.13400

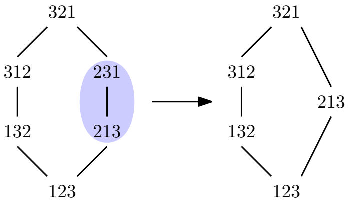
9 February 2023, Combinatorics and Graph Theory Seminar, MSU

Tamari lattice, as quotient of the weak order

$\mathfrak{S}_n$ as a Coxeter group generated by $s_i = (i, i+1)$

For $w \in \mathfrak{S}_n$, $\ell(w) =$ min. length of factorization of w in s_i

Weak order : w covered by w' iff $w' = ws_i$ and $\ell(w') = \ell(w) + 1$



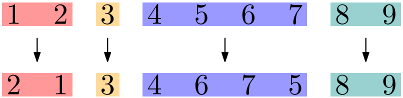
Sylvester class : permutations with the same binary search tree

Only one 231-avoiding in each class. Induced order = Tamari.

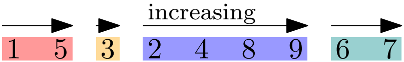
Parabolic subgroup and parabolic quotient of $\mathfrak{S}_n$

Parabolic subgroup : $\langle s_j, j \in J\rangle$ for $J \subseteq [n-1]$

Has the form $\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k}$ with $\alpha = (\alpha_1, \ldots, \alpha_k)$ a composition of n.



Parabolic quotient : $\mathfrak{S}_n^\alpha = \mathfrak{S}_n/(\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k})$.



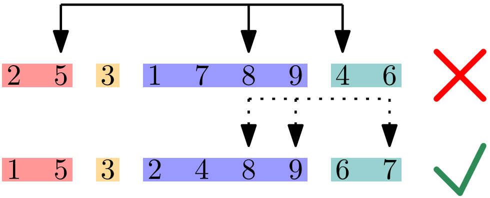
Increasing in each block

Parabolic permutations avoiding 231

$(\alpha, 231)$-pattern : three indices $i < j < k$ in three blocks with

- $w(k) < w(i) < w(j)$,
- $w(k) + 1 = w(i)$.

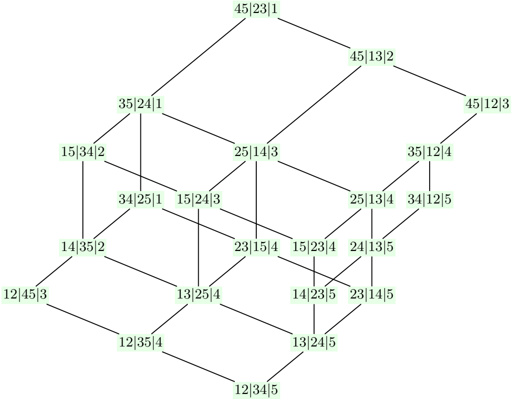
$(\alpha, 231)$-avoiding permutations: without $(\alpha, 231)$ patterns



$\mathfrak{S}_n^\alpha(231)$: set of $(\alpha, 231)$-avoiding permutations

Parabolic Tamari lattice

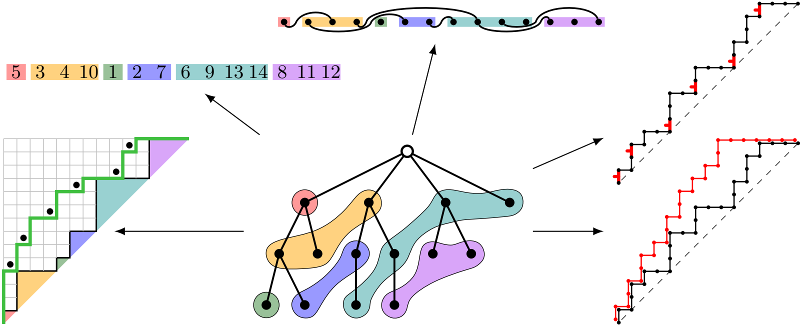
Parabolic Tamari lattice $\mathcal{T}_n^\alpha$ = weak order restricted to $\mathfrak{S}_n^\alpha(231)$ (Mühle–Williams 2019)



Isomorphic to certain ν-Tamari lattices (Ceballos–F.–Mühle 2020).

Parabolic Cataland

Ceballos–F.–Mühle 2020: a world of bijections!



Also related to lattice paths

Recovers the zeta map in q, t-Catalan combinatorics.

Other types?

Coxeter group, type B

Coxeter group: $\langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} \rangle$ with s_i involutions

Classification: $A_n \cong \mathfrak{S}_{n+1}$, B_n, D_n, $I_2(p)$, $E_6, E_7, E_8, F_4, H_3, H_4$

Type B: permutations π of $\pm[n] \stackrel{\text{def}}{=} \{-n, \ldots, -1, 1, \ldots, n\}$ that are sign-symmetric, *i.e.*, $\pi(-i) = -\pi(i)$

One-line notation:

$$\pi = \overline{9}\,\overline{7}\,\overline{8}\,\overline{5}\,\overline{6}\,1\,\overline{3}\,\overline{4}\,2 \mid \overline{2}\,4\,3\,\overline{1}\,6\,5\,8\,7\,9.$$

We may write only the right (positive) part as $\pi = \mid \overline{2}\,4\,3\,\overline{1}\,6\,5\,8\,7\,9$

Also called hyperoctahedral group $\mathfrak{H}_n$

Weak order, type B

Inversion of $\pi \in \mathfrak{H}_n$: indices $i, j \in \pm[n]$ with $i < j$ but $\pi(i) > \pi(j)$

Sign-symmetry $\Rightarrow$ if i, j is an inversion, then $-j, -i$ too.

Thus denoted $(\!(i\ j)\!)$ with $0 < i < j$ or $0 < j < -i$, and $[\![i]\!]$ when $j = -i$

Inversion set of π: set of inversions of π, denoted by $\mathsf{Inv}(\pi)$

Example:

$$\pi = \bar{4}\ \bar{3}\ \bar{5}\ 1\ 2 \mid \bar{2}\ \bar{1}\ 5\ 3\ 4$$

$$\mathsf{Inv}(\pi) = \{[\![1]\!], [\![2]\!], (\!(-1\ 2)\!), (\!(3\ 4)\!), (\!(3\ 5)\!)\}$$

Weak order (left), type B: $\pi \leq_{\mathsf{weak}} \sigma \Leftrightarrow \mathsf{Inv}(\pi) \subseteq \mathsf{Inv}(\sigma)$

Example:

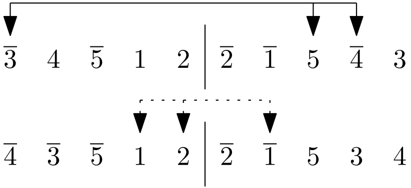
$$\bar{4}\ \bar{5}\ \bar{3}\ \bar{1}\ 2 \mid \bar{2}\ 1\ 3\ 5\ 4 \leq_{\mathsf{weak}} \bar{4}\ \bar{3}\ \bar{5}\ 1\ 2 \mid \bar{2}\ \bar{1}\ 5\ 3\ 4$$

Tamari lattice, type B

Successor in $\pm[n]$: $i^+ = i+1$, except $(-1)^+ = 1$

Type-B 231-pattern in π: indices $i < j < k$ in $\pm[n]$ such that

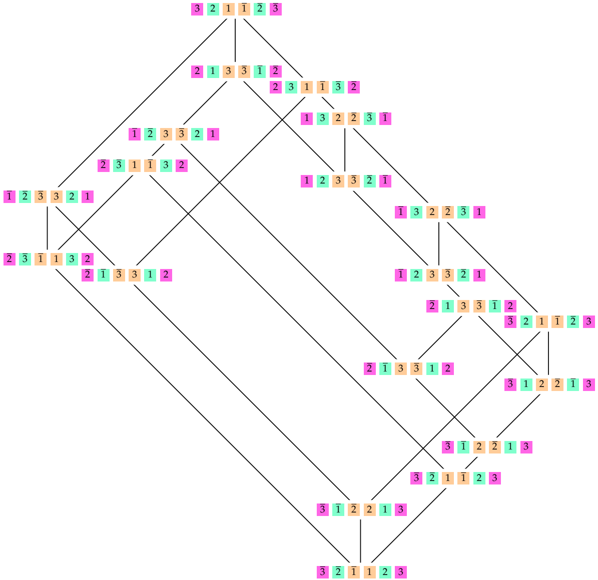
- $j > 0$; (to break sign-symmetry)
- $\pi(j) > \pi(i)$, $\pi(i) = \pi(k)^+$.



231-avoiding sign-symmetric permutations: without type-B 231-pattern

Type-B Tamari lattice (Reading 2007): $\mathrm{Tam}_B(n) \overset{\text{def}}{=} (\mathfrak{H}_n(231), \leq_{\text{weak}})$, with $\mathfrak{H}_n(231)$ the set of type-B 231-avoiding permutations

Example of type-B Tamari lattice



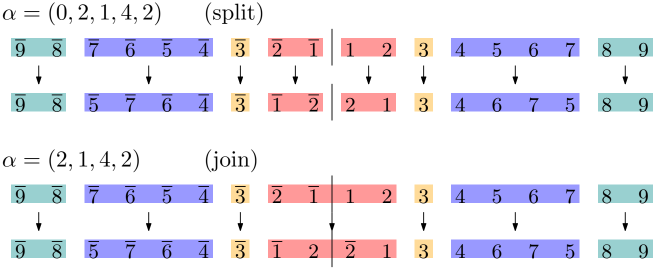
Parabolic subgroup of $\mathfrak{H}_n$

Type-B composition: $\alpha = (\alpha_1, \ldots, \alpha_k)$, with possibly $\alpha_1 = 0$

Generators: $S = \{s_0, s_1, \ldots, s_{n-1}\}$

- For $i \geq 1$, s_i exchanges i and $i+1$ (thus $-i$ and $-i-1$);
- s_0 exchanges 1 and -1.

Parabolic subgroup of $\mathfrak{H}_n$: generated by s_i except for $i = \alpha_1 + \cdots + \alpha_j$



s_0 is special! It makes a difference at the center.

Parabolic quotient of $\mathfrak{H}_n$

Split when α starts with 0, join otherwise.

Parabolic quotient of $\mathfrak{H}_n$, denoted by $\mathfrak{H}_\alpha$

$\alpha = (0, 2, 1, 4, 2)$ (split)

$$\bar{9}\ \bar{8} \quad \bar{3}\ \bar{1}\ 5\ 6 \quad 7 \quad \bar{2}\ 4 \;\big|\; \bar{4}\ 2 \quad \bar{7} \quad \underbrace{\bar{6}\ \bar{5}\ 1\ 3}_{\text{increasing}} \quad 8\ 9$$

$\alpha = (2, 1, 4, 2)$ (join)

$$\bar{9}\ \bar{8} \quad \bar{7}\ \bar{4}\ 1\ 2 \quad 6 \quad \bar{5}\ \bar{3} \,|\, 3\ 5 \quad \bar{6} \quad \underbrace{\bar{2}\ \bar{1}\ 4\ 7}_{\text{increasing}} \quad 8\ 9$$

Regions with lengths determined by α, starting from center

In the join case, the central region is positive for positive indices.

$\mathfrak{H}_\alpha \cong$ interval $[e, \omega_{\mathsf{o};\alpha}]$ in $\mathfrak{H}_n$, with $\omega_{\mathsf{o};\alpha}$ the longest element in $\mathfrak{H}_\alpha$

$$\omega_{\mathsf{o};(0,2,1,4,2)} = 8\ 9 \quad 4\ 5\ 6\ 7 \quad 3 \quad 1\ 2 \quad \bar{2}\ \bar{1} \quad \bar{3} \quad \bar{7}\ \bar{6}\ \bar{5}\ \bar{4} \quad \bar{9}\ \bar{8}$$

$$\omega_{\mathsf{o};(2,1,4,2)} = 8\ 9 \quad 4\ 5\ 6\ 7 \quad 3 \quad \bar{2}\ \bar{1}\ 1\ 2 \quad \bar{3} \quad \bar{7}\ \bar{6}\ \bar{5}\ \bar{4} \quad \bar{9}\ \bar{8}$$

Type-B $(\alpha, 231)$-patterns

Type-B $(\alpha, 231)$-pattern in π: indices $i < j < k$ in $\pm[n]$ such that

- i, j, k in different regions;
- $j > 0$; (to break sign-symmetry)
- $\pi(i) = \pi(k)^+$;
- $\pi(j) > \pi(i)$ when α is split or $j > \alpha_1$; (231)
- $\pi(j) < \pi(k)$ when α is join and $j \leq \alpha_1$. (312)

Split case:

Pattern $4 \mid \bar{7}\ \bar{3} \mid \textcircled{1} \mid \bar{6}\ \bar{2}\ 5 \mid \bar{5}\ 2\ \textcircled{6} \mid \textcircled{$\bar{1}$} \mid 3\ 7 \mid \bar{4}$

Pattern $4 \mid \bar{5}\ \bar{1} \mid 6 \mid \bar{2}\ 3\ 7 \mid \bar{7}\ \textcircled{$\bar{3}$}\ 2 \mid \bar{6} \mid 1\ \textcircled{5} \mid \textcircled{$\bar{4}$}$

Join case:

Not pattern $\textcircled{$\bar{6}$}\ \bar{4} \mid \bar{8}\ 7 \mid \bar{5}\ \bar{3}\ \bar{2}\ \bar{1}\ \textcircled{1}\ 2\ 3\ 5 \mid \textcircled{$\bar{7}$}\ 8 \mid 4\ 6$

Pattern $\bar{7}\ \textcircled{5} \mid \bar{4}\ 3 \mid \bar{8}\ \bar{6}\ \bar{2}\ \bar{1}\ 1\ \textcircled{2}\ 6\ 8 \mid \bar{3}\ \textcircled{4} \mid \bar{5}\ 7$

Flipped for the joined region!

Type-B $(\alpha, 231)$-avoiding permutations

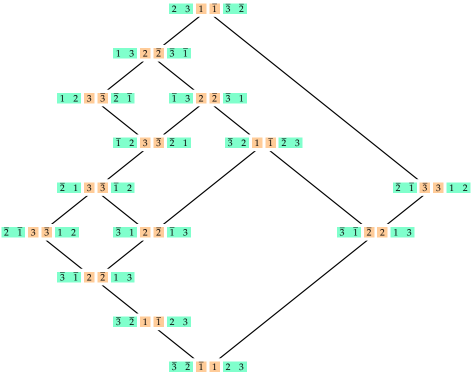
Type-B $(\alpha, 231)$-avoiding permutations: $\pi \in \mathfrak{H}_\alpha$ without such pattern

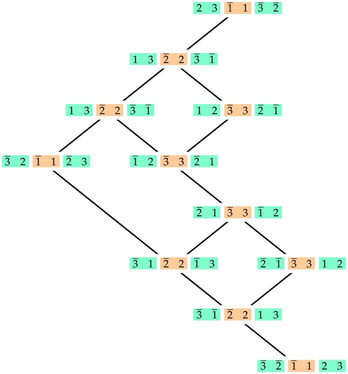
Split case	$\overline{3}$ $\overline{1}$ 5 6 | 7 | $\overline{2}$ 4 | $\overline{4}$ 2 | $\overline{7}$ | $\overline{6}$ $\overline{5}$ 1 3
Join case	$\overline{7}$ $\overline{4}$ 1 2 | 6 | $\overline{5}$ $\overline{3}$ 3 5 | $\overline{6}$ | $\overline{2}$ $\overline{1}$ 4 7

$\mathfrak{H}_\alpha(231)$: the set of type-B $(\alpha, 231)$-avoiding permutations

Type-B parabolic Tamari lattice

Type-B parabolic Tamari lattice: $\mathsf{Tam}_B(\alpha) = (\mathfrak{H}_\alpha, \leq_{\mathsf{weak}})$





Type-B Parabolic Tamari as quotient lattice

Everything just like the classical Tamari lattice!

Theorem (F.–Mühle-Novelli 2023+)

For any type-B composition α, $\mathsf{Tam}_B(\alpha)$ is a lattice. Moreover, it is a quotient lattice of the weak order of $\mathfrak{H}_\alpha$.

Congruence classes defined by downward projection $\Pi_\downarrow$:

1. For each $(\alpha, 231)$-pattern i, j, k, exchanges $\pi(i)$ and $\pi(k)$.
2. Repeat 1 until no such pattern exists.

Gives the smallest element in the class

Also upward projection $\Pi_\uparrow$ using $(\alpha, 312)$-avoiding permutations, giving the largest element.

Two projections are compatible and preserve the weak order. The class is the interval in between.

Lattice properties

Theorem (F.–Mühle-Novelli 2023+)

For any type-B composition α, $\mathsf{Tam}_B(\alpha)$ is a congruence uniform (thus semi-distributive) and trim.

Proof:

- Congruence uniform: quotient lattice of $\mathfrak{H}_n$
- Semi-distributive: from congruence uniform
- Extremal: explicit counting of length and join-irreducibles

$$\omega_{\mathsf{o};(0,2,1,4,2)} = 8\ 9\ 4\ 5\ 6\ 7\ 3\ 1\ 2\ \overline{2}\ \overline{1}\ \overline{3}\ \overline{7}\ \overline{6}\ \overline{5}\ \overline{4}\ \overline{9}\ \overline{8}$$

$$\omega_{\mathsf{o};(2,1,4,2)} = 8\ 9\ 4\ 5\ 6\ 7\ 3\ \overline{2}\ \overline{1}\ 1\ 2\ \overline{3}\ \overline{7}\ \overline{6}\ \overline{5}\ \overline{4}\ \overline{9}\ \overline{8}$$

$$|\mathsf{Inv}(\omega_{\mathsf{o};\alpha})| = n^2 - \sum_i \binom{\alpha_i}{2} - \binom{\alpha_2+1}{2}[\alpha_1 \neq 0].$$

- Trim: from extremal and semi-distributive

Where are the conditions from?

Reading 2007: Universal construction of Tamari (Cambrian) lattices for all type

On c-aligned elements, with c a Coxeter element (product of all s_i)

Type B: we take $c = s_0 s_1 \cdots s_{n-1}$

π is c-aligned $\Leftrightarrow$ forcing relations: some $t \in \mathsf{Cov}(\pi) \Rightarrow$ some $s \in \mathsf{Inv}(\pi)$

Determined by a linear order of inversions given by the c-sorting word of the longest element in $\mathfrak{H}_n$

Type B, **parabolic**: replace the longest element in $\mathfrak{H}_n$ by that in $\mathfrak{H}_\alpha$

A slide not meant to be read

!!! Headache warning !!!

$\pi \in \mathfrak{H}_\alpha$ is c-aligned if, for all $1 \leq i < k \leq n$,

(1) if $[\![i]\!] \in \mathrm{Cov}(\pi)$, then $[\![j]\!] \in \mathrm{Inv}(\pi)$ for all $1 \leq j < i$ with i, j in different regions;

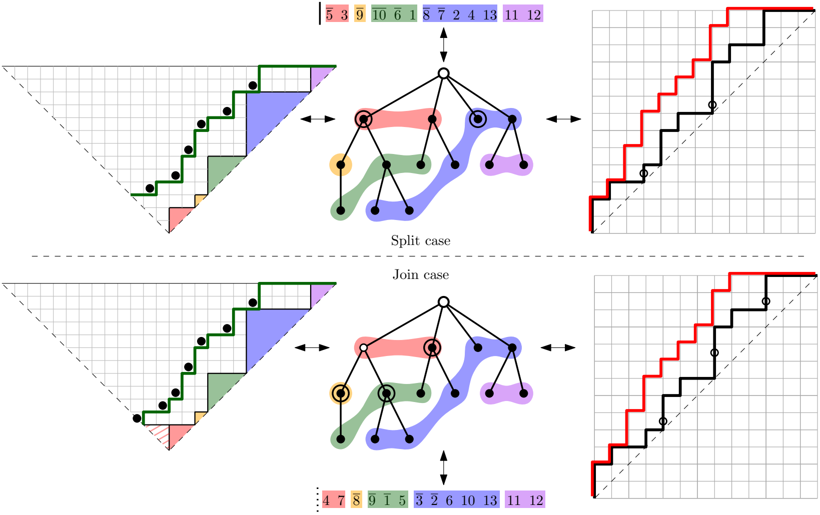
(2) if $(\!(i\ k)\!) \in \mathrm{Cov}(\pi)$, then $(\!(i\ j)\!) \in \mathrm{Inv}(\pi)$ such that i, j, k are in different regions;

(3) if $(\!(-k\ i)\!) \in \mathrm{Cov}(\pi)$, then

- (3a) $[\![i]\!] \in \mathrm{Inv}(\pi)$ when $i > \alpha_1$ or α is split,
- (3b) $(\!(-j\ i)\!) \in \mathrm{Inv}(\pi)$ for $1 \leq j < k$ with j, k in different regions when α is split or $j > \alpha_1$,
- (3c) $(\!(j\ k)\!) \in \mathrm{Inv}(\pi)$ when $j \leq \alpha_1$, $j \neq i$ and α is join,
- (3d) $(\!(-k\ j)\!) \in \mathrm{Inv}(\pi)$ for $1 \leq j < i$ with i, j in different regions when α is split or $j > \alpha_1$,
- (3e) $(\!(j\ i)\!) \in \mathrm{Inv}(\pi)$ when $i > j > \alpha_1$ and α is join.

Summed up nicely by pattern avoidance !

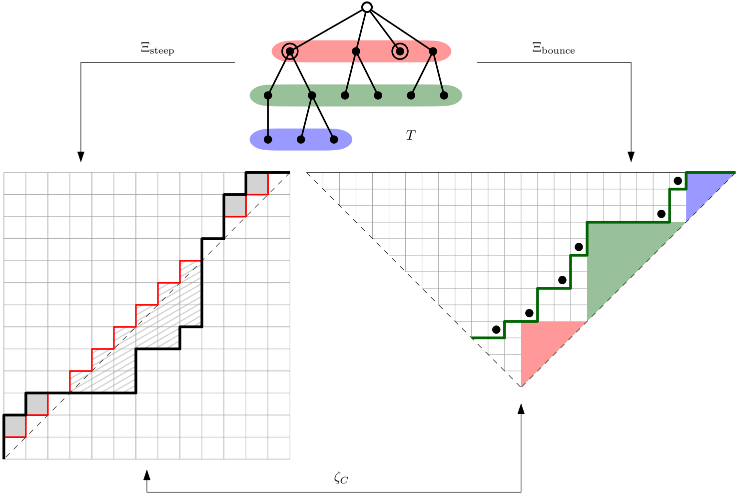
Combinatorial models



Work in progress. Some bijections clear, some with multiple choices.

Type-C zeta map

Sulzgruber–Thiel 2018: (labelled) Zeta map for type B, C and D



We recover (labelled) zeta map for type C. Also transfer dinv $\leftrightarrow$ area.

Some enumerative consequences

Cover inversion of π: inversion $(\!(i\ j)\!)$ (or $[\![i]\!]$) with $\pi(i) = \pi(j)^+$.

$\mathsf{Cov}(\pi)$: the set of cover inversions of π.

Proposition (F.–Mühle–Novelli 2023+)

Take $c(\alpha) = \sum_{\pi \in \mathfrak{H}_\alpha(231)} x^{|\mathsf{Cov}(\pi)|}$. *Then for* $\alpha = (t, 1, \ldots, 1)$, *we have*

$$c(\alpha) = \sum_{k=0}^{n-t} \binom{n-t}{k}\binom{n+t}{k} x^k.$$

Thus $|\mathfrak{H}_\alpha(231)| = \binom{2n}{n-t}$.

Proposition (F.–Mühle–Novelli 2023+)

For $\alpha = (0, 1, 1, \ldots, 1, 2)$, $|\mathfrak{H}_\alpha(231)|$ *is the type-D Catalan number:*

$$|\mathfrak{H}_\alpha(231)| = \frac{3n-2}{n}\binom{2n-2}{n-1}.$$

What we are doing now

- Combinatorial description of the order?
- Link to possible type-B ν-Tamari (Ceballos–Padrol–Sarmiento '19)?
- Type-B q, t-Catalan statistics (Stump 2010)?
- Enumeration? (Lattice path model known for split case)

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Thank you for your attention!