An New Shifted Littlewood–Richardson Rule

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February 7, 2024

Outline

Symmetric Functions

A *symmetric function* is a member of the ring $R[[x_1, x_2, \ldots]]$ of formal power series of bounded degree over countably infinite indeterminates, invariant under permutations of its subscripts. Famous bases of the symmetric functions include:

- The *homogeneous symmetric functions*:

$$h_n := \sum_{i_1 \leq i_2 \leq \ldots \leq i_n} x_{i_1} \ldots x_{i_n}.$$

- The *elementary symmetric functions*:

$$e_n := \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} \ldots x_{i_n}$$

- The *power sum symmetric functions*:

$$p_n := \sum_i x_i^n$$

Symmetric Functions

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, and $f \in Sym$, we let $f_\lambda = f_{(\lambda_1, \ldots, \lambda_k)}$ signify:

$$f_\lambda = f_{\lambda_1} \ldots f_{\lambda_k}.$$

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There is yet another basis for *Sym*, specially relevant due to its connection to the representation theory of the symmetric group; namely, that of *Schur functions*:

$$s_{(3,2)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3x_2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

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These monomials correspond to the tableaux:

1	1
2	

(a) Tableau for $x_1^2x_2$.

1	1
3	

(b) Tableau for $x_1^2x_3$.

1	2
2	

(c) Tableau for $x_1x_2^2$.

1	2
3	

(d) Tableau for $x_1x_2x_3$.

1	3
2	

(e) Tableau for $x_1x_3x_2$.

1	3
3	

(f) Tableau for $x_1x_3^2$.

2	2
3	

(g) Tableau for $x_2^2x_3$.

2	3
3	

(h) Tableau for $x_2x_3^2$.

Combinatorial Link between P_λ and s_λ

Analogous to the combinatorial definition of s_λ, we have that

$$P_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + 2(x_1x_2x_3) + x_1x_2^2 + x_2^2x_3 + x_2x_3^2$$

1	1
	2

(a) Tableau for $x_1^2x_2$.

1	1
	3

(b) Tableau for $x_1^2x_3$.

1	2
	3

(c) Tableau for $x_1x_2x_3$.

1	$2'$
	3

(d) Tableau for $x_1x_2x_3$.

1	$2'$
	2

(e) Tableau for $x_1x_2^2$.

2	2
	3

(f) Tableau for $x_1x_3^2$.

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	3

(g) Tableau for $x_2x_3^2$.

Back to Symmetric Functions

We want to count $c^{\nu}_{\lambda,\mu}$, the structure coefficients in

$$s_\lambda \cdot s_\mu = \sum_{\nu} c^{\nu}_{\lambda,\mu} s_\nu .$$

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$$s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda,\mu} s_\nu .$$

An algorithm constructed by Schensted paves the way for the solution of this problem.

Schensted's Insertion Algorithm

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$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftarrow 1 \quad = \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftarrow 2 \quad = \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

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Theorem (Schensted 1961)

Let $\pi \in \mathcal{S}_n$ be a permutation. Then the longest increasing (decreasing) subsequence of π is of length equal to the length of the first row (column) of $P(\pi)$.

The Plactic Monoid

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  For instance, $[4213] \cdot [65]$ can be calculated as follows:

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \leftarrow 6 \leftarrow 5 \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \leftarrow 5 \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 6 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$$

Plactic Littlewood–Richardson Rule

Theorem (Schützenberger 1977)

The Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu}$ is equal to the number of pairs (T_μ, T_ν) such that

$$T_\mu \cdot T_\nu = T_\lambda$$

for a fixed plactic class T_λ of shape λ.

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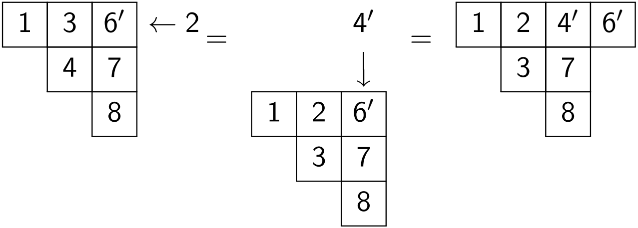

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$$\begin{array}{|c|c|c|} \hline 1 & 3 & 6' \\ \hline \multicolumn{1}{c|}{} & 4 & 7 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 \\ \cline{3-3} \end{array} \leftarrow 2 = \begin{array}{ccc} & & 4' \\ & & \downarrow \\ 1 & 2 & 6' \\ & 3 & 7 \\ & & 8 \end{array} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4' & 6' \\ \hline \multicolumn{1}{c|}{} & 3 & 7 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}$$

Theorem (Sagan, Serrano)

The length of the longest hook subword of a word w is equal to the length of the top row of $P_{mix(w)}$.

The Shifted Plactic Monoid and Representation Theory

Theorem (Serrano 2009)

The shifted Littlewood-Richardson coefficient $f^{\lambda}_{\mu,\nu}$ is equal to the number of pairs (T_μ, T_ν) such that

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for a fixed shifted plactic class T_λ of shape λ.

Plactic Littlewood–Richardson Rule

Recalling Schensted's result and its generalizations, given a shape λ, for instance $\lambda = (4, 3, 1)$ the word

$$3\ 222\ 1111$$

with content equal to λ inserts to T of shape λ.

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This is from that perspective the most natural candidate for that shape. Let $T_\lambda = T$. Then, if

$$T_\mu \cdot T_\nu = T_\lambda$$

we also have that

$$T_\nu = P(\ \ell(\nu)^{\nu_{\ell(\nu)}} \qquad (\ell(\nu)-1)^{\nu_{\ell(\nu)-1}} \qquad \dots \qquad 1^{\nu_1}\ )$$

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If T_λ is Yamanouchi, it follows that T_ν is Yamanouchi too. Then $c^{\lambda}_{\mu,\nu}$ is equal to the number of tableaux T_μ such that

$$T_\mu \cdot Y_\nu = Y_\lambda$$

Motivation

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It does not have the right factor property, as

$$P_{\text{mix}}(32221111) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2' & 3' \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{3}{c}{} \\ \cline{2-3} \end{array}$$
$$= P_{\text{mix}}(3211) \cdot P_{\text{mix}}(2211)$$

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$$= P_{\text{mix}}(3211) \cdot P_{\text{mix}}(2211)$$

But 2211 is not a word with partition content. So certain classes that project to the Yamanouchi class in the plactic monoid do not have the right factor property!

New Yamanouchi Analogues

Definition (Serrano):

Let $w = w_1 w_2 \ldots w_n \in \mathbb{N}^*_{>0}$ be a word. We say that it is a *hook word* if there exists $1 \leq k \leq n$ such that

$$w_1 > w_2 > \ldots > w_k \leq w_{k+1} \leq w_{k+2} \leq \ldots \leq w_n$$

where k is possibly 1 or n.

Theorem (Serrano):

Let $w \in \mathbb{N}^*_{>0}$ and $T = P_{\text{mix}}(w)$ be a shifted tableau. Then, the shape λ of T is completely determined by the tuple $(I_1(w), I_2(w), \ldots, I_{\ell(\lambda)}(w))$. Furthermore,

$$I_k(w) = \lambda_1 + \ldots + \lambda_k + \binom{k}{2}.$$

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This is an analogue of Schensted's theorem and Greene's generalization for the shifted context. So if we can find a word that generalizes the increasing (decreasing) properties of Yamanouchi words for the shifted setting we have a good candidate!

Barely Yamanouchi Words

Let again $\lambda = (7, 3, 1)$ and consider

$$1\,321\,7654321.$$

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Then

$P_{\text{mix}}(\hat{y}_{(4,3,1)}) =$

1	1	1	3′	5′	6′	7′
	2	2	4′			
		3				

$$\hat{y}_\nu := (\nu_{\ell(\nu)})(\nu_{\ell(\nu)}-1)\cdots 1(\nu_{\ell(\nu)-1})(\nu_{\ell(\nu)-1}-1)\cdots 1 \cdots (\nu_1)(\nu_1-1)\cdots 1$$

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A word $w \in \mathbb{N}^*_{>0}$ is barely Yamanouchi if and only if as we read from right to left either of these two conditions:

- $i(w) = i(w+1)$ or
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Note that all of these words are also Yamanouchi!

Examples

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All barely Yamanouchi words of the same content, with the same number of primed entries insert to the same tableaux.

A Shifted Plactic LR–Rule

Recall the LR–rule for the shifted plactic monoid:

Theorem (Serrano 2009)

The shifted Littlewood-Richardson coefficient $f^{\lambda}_{\mu,\nu}$ is equal to the number of pairs (T_μ, T_ν) such that

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Since barely Yamanouchi words have the right factor property, it is enough to count the tableaux of shape μ such that

$$T_\mu \cdot \hat{Y}_\nu = \hat{Y}_\lambda$$

Combinatorial Description of Left Factors

A word that can be realized as the left factor of a barely Yamanouchi word is called *scarcely Yamanouchi*.

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Moreover, we say that a barely Yamanouchi word w is *interlacing* if whenever i is in the j–th shrinking sequence of w, it is to the left of $i+1$ in the $j-1$–th shrinking sequence of w.

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Moreover, we say that a barely Yamanouchi word w is *interlacing* if whenever i is in the j–th shrinking sequence of w, it is to the left of $i+1$ in the $j-1$–th shrinking sequence of w.
Interlacing words have a compact combinatorial description.

Example of an Interlacing Tableau

Let $\nu = (10, 9, 5)$, $\lambda = (9, 6, 4)$, and $\mu = (3, 2)$.
Then,

$$\bigcup_{i=1}^{\ell(\nu)} (\mu_i, \nu_i] \cap \mathbb{Z} = \{10, 9, 8, 7, 6, 5, 4\} \cup \{9, 8, 7, 6, 5, 4, 3\} \cup \{5, 4, 3, 2, 1\}$$

Interlacing tableaux can be constructed by steps, laying down entries from each segment first forming an unprimed vertical strip and then a horizontal primed strip.

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	4	$5'$	$6'$	$7'$				

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1	2′	3′	3	4′	5′	8′	9′	10′
	4	4	5′	6′	7′	9′		
		5	6′	7′	8′			

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Lemma (E.-Pechenik)

If the canonical word $\hat{y}_\nu$ is fixed as a right factor, then the left factor in the product

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Main Theorem

Theorem (E.-Pechenik):

The coefficient $b^{\nu}_{\lambda,\mu}$ is equal to the tableaux T_λ on the set of letters $\bigcup_{i=1}^{\ell(\nu)} (\mu_i, \nu_i] \cap \mathbb{N}$ such that:

- For every j, the letters of $(\mu_j, \nu_j] \cap \mathbb{N}$ in T_λ consist of a vertical strip of unprimed letters increasing downwards, and a horizontal strip of primed letters increasing from left to right, all of whose entries are on columns greater than those containing the vertical strip.
- The unprimed entries of $(\mu_j, \nu_j] \cap \mathbb{N}$ occur before the unprimed entries of $(\mu_k, \nu_k] \cap \mathbb{N}$ for all $k < j$ when on the same row.
- The primed entries of $(\mu_j, \nu_j] \cap \mathbb{N}$ occur before the primed entries of $(\mu_k, \nu_k] \cap \mathbb{N}$ for all $k < j$ when on the same column.
- None of the sequences can be extended in T_λ.

Example

Let $\nu = (5,3,2)$, $\lambda = (3,1)$, and $\mu = (4,2)$. Then, $\bigcup_{i=1}^{\ell(\nu)} (\mu_i, \nu_i] \cap \mathbb{Z} = \{5\} \cup \{3\} \cup \{2,1\}$, and there are 4 tableaux that can be constructed from these letters by placing each segment as a union of a vertical strip of unprimed letters and a horizontal strip of primed letters:

1	$2'$	3
	5	

(a) First valid tableau.

1	$2'$	5
	3	

(b) Second valid tableau.

1	$2'$	$5'$
	3	

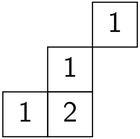
(c) Third valid tableau.

1	$2'$	$3'$
	$5'$	

(d) Invalid tableau.

Example

For the same partitions λ, μ, and ν as in Example 4, Stembridge's rule requires the consideration of 8 tableaux as shown below.



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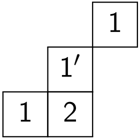
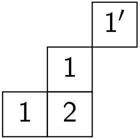


(b) Second valid tableau.



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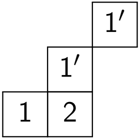

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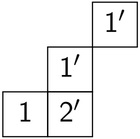
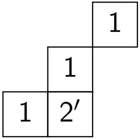
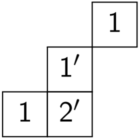


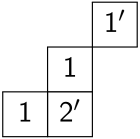
(e) Invalid tableau.



(f) Invalid tableau.



(g) Invalid tableau.



(h) Invalid tableau.

Example

To better illustrate the condition that the sequences of the tableaux should be such that none of them can be extended, further consider the invalid tableau.

1	$2'$	$3'$	3	$4'$	$5'$	$8'$	$9'$	$10'$
	4	4	$5'$	$6'$	$7'$	$9'$		
		5	$6'$	$7'$	$8'$			

$\rightarrow$

1	$2'$	$3'$	3	$4'$	$5'$	$8'$	$9'$	$10'$
	4	4	$5'$	$6'$	$7'$	$9'$		
		5	$6'$	$7'$	$8'$			

The End

Thank you!