Descents on quasi-Stirling permutations

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Descents

Definition

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Example

$\mathrm{des}(36522131) = 5$

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These polynomials appear in work of Euler from 1755.

Eulerian polynomials

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2\,(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1.2.3\,(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1.2.3.4\,(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1.2.3.4.5\,(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1.2.3.4.5.6\,(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1.2.3.4.5.6.7\,(p-1)^7}$$

&c.

L. Euler, 1755.

Eulerian Polynomials

$$\frac{A_n(p)/p}{n!(p-1)^n} \quad (1 \le n \le 7)$$

Eulerian polynomials

Euler was considering the series

$$\sum_{m\geq 0} m t^m = \frac{t}{(1-t)^2}$$

$$\sum_{m\geq 0} m^2 t^m = \frac{t+t^2}{(1-t)^3}$$

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In general,

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}.$$

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What are the polynomials in the numerator?

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We have $|\mathcal{Q}_n| = (2n-1)!! = (2n-1)\cdot(2n-3)\cdot\dots\cdot 3\cdot 1$, since every permutation in $\mathcal{Q}_n$ can be obtained by inserting nn into one of the $2n-1$ spaces of a permutation in $\mathcal{Q}_{n-1}$.

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Theorem (Gessel–Stanley '78)

$$\sum_{m \geq 0} S(m+n, m)\, t^m = \frac{Q_n(t)}{(1-t)^{2n+1}}.$$

Literature on Stirling permutations

There is an extensive literature on Stirling permutations. Some work relevant to this talk:

- Bóna '08: $Q_n(t)$ also gives the enumeration of $\mathcal{Q}_n$ by the number of plateaus, that is, positions i such that $\pi_i = \pi_{i+1}$.

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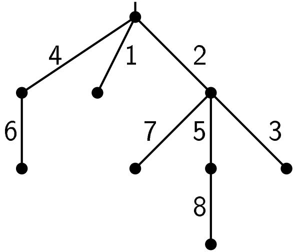
- Janson '08: The joint distribution of ascents, descents and plateaus on $\mathcal{Q}_n$ is asymptotically normal.
- The coefficients of $Q_n(t)$ are sometimes called second-order Eulerian numbers.

Stirling permutations and trees

$\mathcal{I}_n$ = set of increasing edge-labeled plane rooted trees with n edges.



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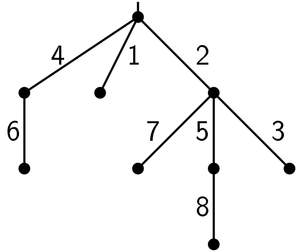

$\xrightarrow{\varphi}$ 4664112775885332

Theorem (Koganov '96, Janson '08)

There is a bijection $\varphi : \mathcal{I}_n \longrightarrow \mathcal{Q}_n$ obtained by traversing the edges of the tree along a depth-first walk from left to right, and recording their labels.

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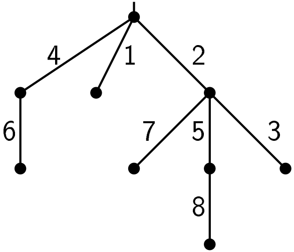

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If we remove the increasing condition on the trees, what is the image of φ?

Quasi-Stirling permutations and trees

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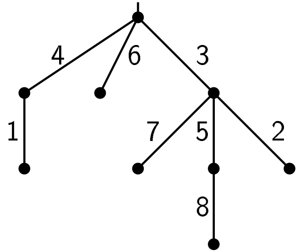

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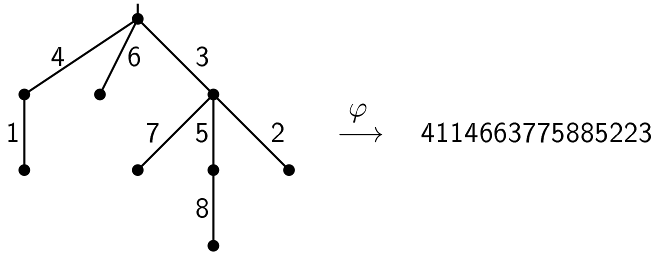

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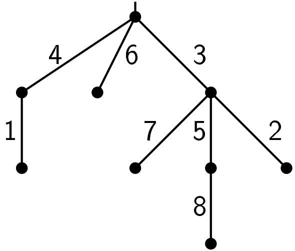

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Definition (Archer–Gregory–Pennington–Slayden '19)

A quasi-Stirling permutation is a permutation of the multiset $\{1,1,2,2,\dots,n,n\}$ that avoids the patterns 1212 and 2121.

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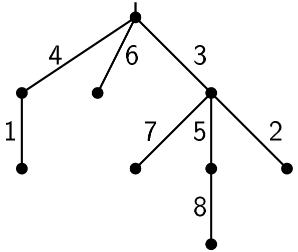

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In other words, it does not have four positions $i < j < k < \ell$ with $\pi_i = \pi_k$ and $\pi_j = \pi_\ell$.

Quasi-Stirling permutations

$\overline{\mathcal{Q}}_n$ = set of quasi-Stirling permutations of $\{1, 1, 2, 2, \ldots, n, n\}$.

Example

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It follows that

$$|\overline{\mathcal{Q}}_n| = n!\,C_n = \frac{(2n)!}{(n+1)!}.$$

Descents on quasi-Stirling permutations

Conjecture (Archer–Gregory–Pennington–Slayden '19)

The number of $\pi \in \overline{\mathcal{Q}}_n$ with $\mathrm{des}(\pi) = n$ is equal to $(n+1)^{n-1}$.

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Set of $\pi \in \overline{\mathcal{Q}}_3$ with $\operatorname{des}(\pi) = 1$: $\{112233\}$ 1

with $\operatorname{des}(\pi) = 2$: 13
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with $\operatorname{des}(\pi) = 3$: 16
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To prove this conjecture, we look at how descents are transformed by the bijection φ.

Descents on quasi-Stirling permutations

Lemma

If $T \in \mathcal{T}_n$ and $\pi = \varphi(T) \in \overline{\mathcal{Q}}_n$, then

$$\operatorname{des}(\pi) = \operatorname{cdes}(T),$$

where $\operatorname{cdes}(T)$ *is obtained by adding the number of cyclic descents of the edge labels counterclockwise around each vertex of* T.

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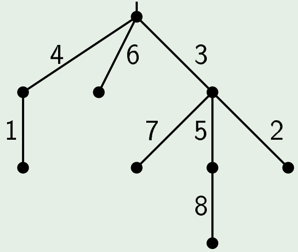

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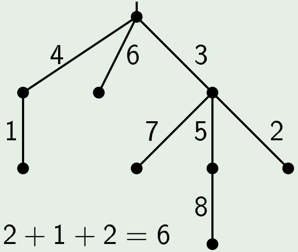

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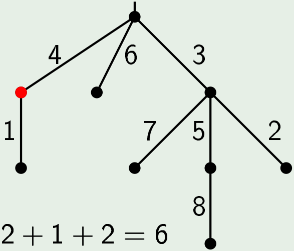

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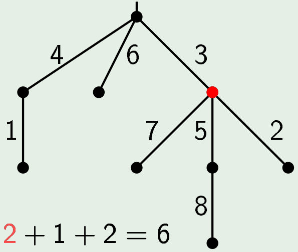

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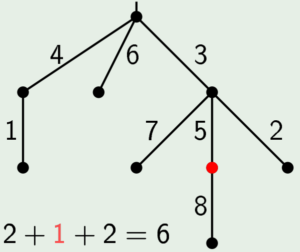

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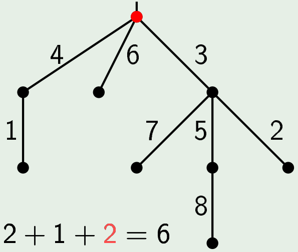

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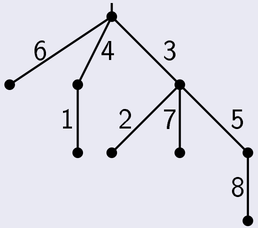
Proof sketch.

Equivalent to counting $T \in \mathcal{T}_n$ with $\mathrm{cdes}(T) = n$, i.e., trees where the number of cyclic descents around each vertex is maximized.



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The number of $\pi \in \overline{\mathcal{Q}}_n$ *with* $\mathrm{des}(\pi) = n$ *is equal to* $(n+1)^{n-1}$.

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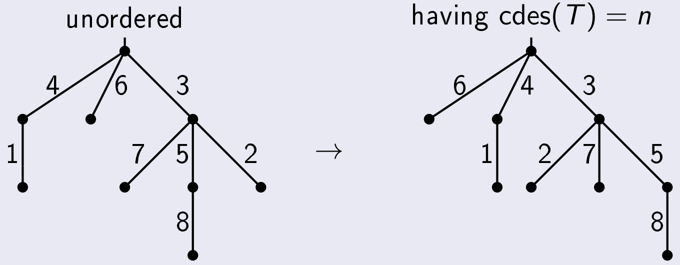
Equivalent to counting $T \in \mathcal{T}_n$ with $\mathrm{cdes}(T) = n$, i.e., trees where the number of cyclic descents around each vertex is maximized.
Such trees are in bijection with *unordered* trees:



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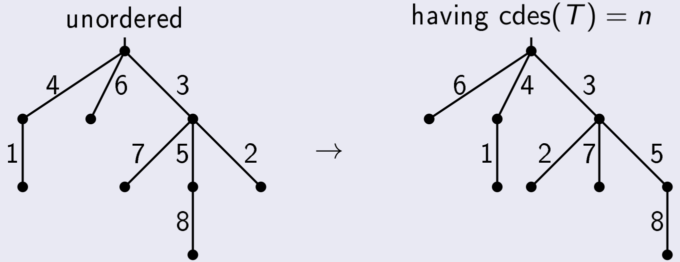

By Cayley's formula, there are $(n+1)^{n-1}$ such trees. □

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Define their exponential generating function (EGF):

$$\overline{Q}(t,z) = \sum_{n\geq 0} \overline{Q}_n(t) \frac{z^n}{n!}.$$

EGF for Eulerian polynomials

Recall the Eulerian polynomials

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Now we are ready to give an expression for $\overline{Q}(t,z)$.

Descents on quasi-Stirling permutations

Theorem

The EGF $\overline{Q}(t,z)$ *for quasi-Stirling permutations by the number of descents satisfies the implicit equation*

$$\overline{Q}(t,z) = A(t, z\overline{Q}(t,z)),$$

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$$\overline{Q}_n(t) = \frac{n!}{n+1}\,[z^n]A(t,z)^{n+1}.$$

Here $[z^n]F(z)$ denotes the coefficient of z^n in $F(z)$.

Proof ideas

By the bijection φ,

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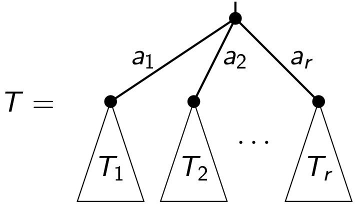

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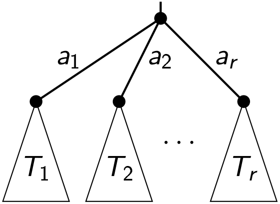
$T =$ 

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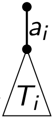
$$\mathrm{cdes}(T) = \sum_{i=1}^{r} (\mathrm{cdes}($$ 

$$) - 1) + \mathrm{des}(a_1 a_2 \dots a_r).$$

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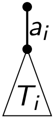
The EGF for each piece 

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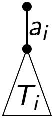
The EGF for each piece 

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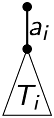
The EGF for each piece 

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Recall the formulas:

$$\sum_{m\geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}} \qquad \text{(Eulerian)}$$

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Open: Find a combinatorial proof.

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Recall: i is a plateau of π if $\pi_i = \pi_{i+1}$,
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The roots of the Eulerian polynomials $A_n(t)$ are real, distinct, and nonpositive.

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- *The coefficients of $\overline{Q}_n(t)$ are unimodal and log-concave.*
- *The distribution of the number of descents on $\overline{\mathcal{Q}}_n$ converges to a normal distribution as $n \to \infty$.*

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Proving real-rootedness of $\overline{Q}_n(t)$ is more complicated than for $A_n(t)$ or $Q_n(t)$, because for quasi-Stirling permutations there is no simple recursive description relating $\overline{\mathcal{Q}}_n$ and $\overline{\mathcal{Q}}_{n-1}$.

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For $k = 1$, $\mathcal{Q}_n^1 = \overline{\mathcal{Q}}_n^1 = \mathcal{S}_n$. For $k = 2$, $\mathcal{Q}_n^2 = \mathcal{Q}_n$ and $\overline{\mathcal{Q}}_n^2 = \overline{\mathcal{Q}}_n$.

Enumeration of k-Stirling and k-quasi-Stirling permutations

Counting k-Stirling permutations is easy, since every permutation in $\mathcal{Q}_n^k$ can be obtained by inserting the string $n^k = nn\dots n$ into one of the $(n-1)k+1$ spaces of a permutation in $\mathcal{Q}_{n-1}^k$, so

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Theorem

For $n \geq 1$ and $k \geq 1$,

$$|\overline{\mathcal{Q}}_n^k| = \frac{(kn)!}{((k-1)n+1)!} = n!\, C_{n,k},$$

where

$$C_{n,k} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

is the nth k-Catalan number.

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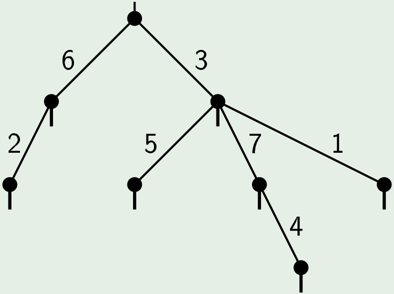
Example

A bijection between *compartmented trees* and 3-quasi-Stirling permutations:



$\xrightarrow{\phi}$ 622266355537744471113

Ascents, descents and plateaus on k-quasi-Stirling permutations

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Define the multivariate k-quasi-Stirling polynomials

$$\overline{P}_n^{(k)}(q,t,u) = \sum_{\pi \in \overline{\mathcal{Q}}_n^k} q^{\mathrm{asc}(\pi)} t^{\mathrm{des}(\pi)} u^{\mathrm{plat}(\pi)},$$

and their EGF

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The proof follows ascents, descents and plateaus through the bijection ϕ, and it uses a decomposition of compartmented trees.