On the f-vector of flow polytopes for complete graphs

MSU Combinatorics and Graph Theory Seminar

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October 9, 2024

Outline

Motivation

- The Chan-Robbins-Yuen polytope (CRY_n) of order n is defined as the convex hull of n-by-n permutation matrices π for which $\pi_{i,j} = 0$ for $j \geq i + 2$.

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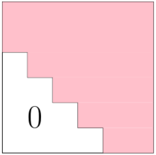

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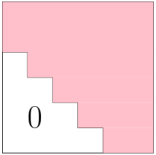

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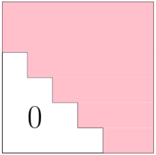

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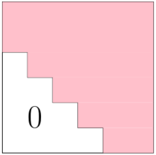

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- Face of the Birkhoff polytope of doubly stochastic matrices having dimension $\binom{n}{2}$ and 2^{n-1} vertices.
- CRY_n is also an example of a flow polytope over the complete graph...

Section 1

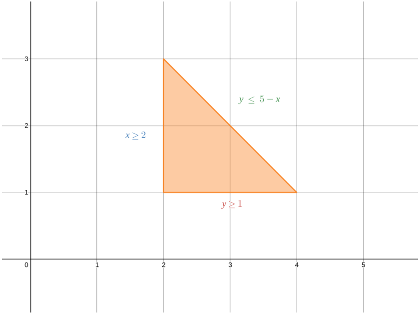
Flow Polytopes

Integral (Convex) Polytopes

Definition

A compact subset $P \subseteq \mathbb{R}^n$ is a **convex polytope** if P can be written as

1. the intersection of finitely many half spaces, or
2. the convex hull of finitely many vertices



f-vectors

- The f-**polynomial** (or f-**vector**) of a polytope P is

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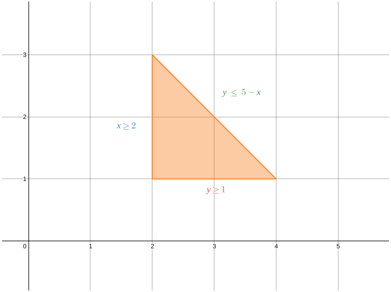

- $f_P(x) = \frac{1}{x} + 3 + 3x^1 + x^2.$

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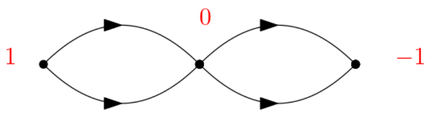

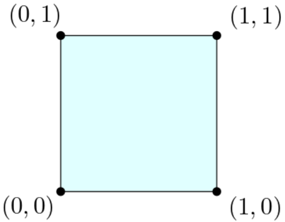

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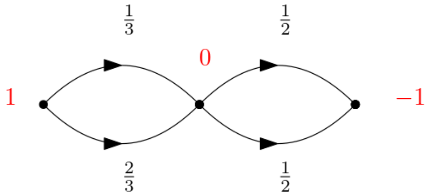

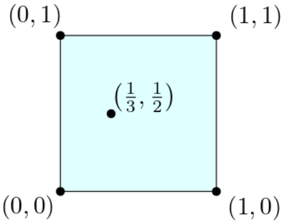

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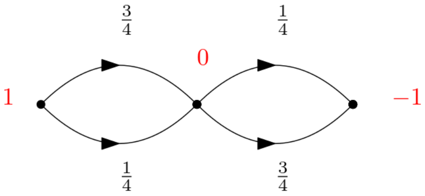

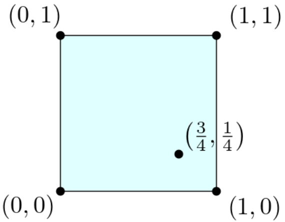

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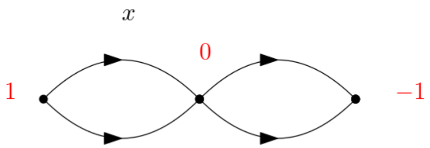

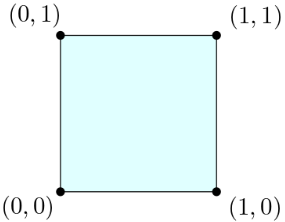

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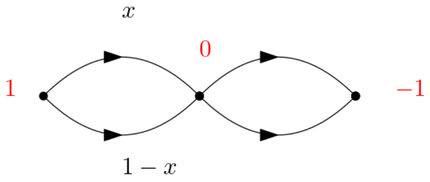

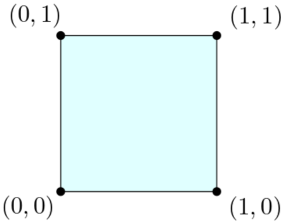

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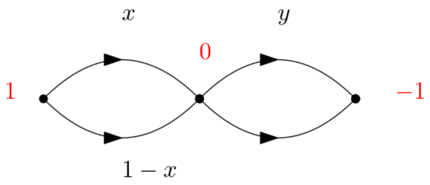

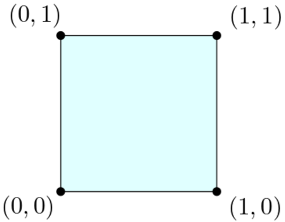

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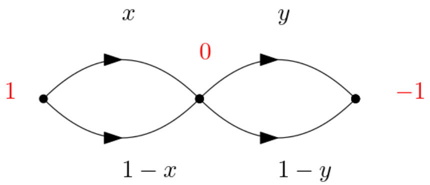

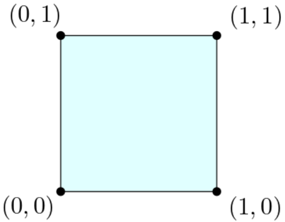

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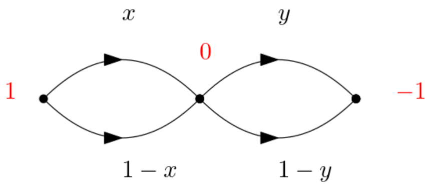

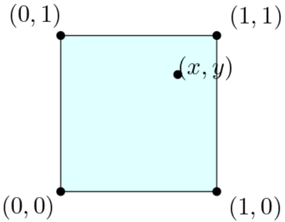

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$$x_{12} + x_{13} + x_{14} = a_1$$
$$x_{23} + x_{24} - x_{12} = a_2$$
$$x_{34} - x_{13} - x_{23} = a_3$$

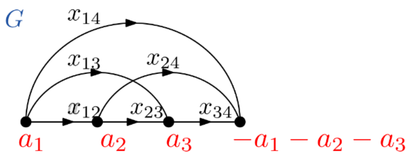

Figure: The complete graph K_4 and the hyperplanes bounding $\mathbf{Flow}_3(a_1, a_2, a_3)$

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- The f-vector for other cases is harder to compute.
- $\mathbf{Flow}_n(1, 1, \ldots, 1)$ is simple, whereas general instances of $\mathbf{Flow}_n(\mathbf{a})$ (including the case of CRY_n) are not.

Main Goal

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To compute the f-vector of $\mathbf{Flow}_n(\mathbf{a})$ for arbitrary (nonnegative) $\mathbf{a} \in \mathbb{N}^n$.

Some Data

n	f-vector of CRY_n
1	1, 1
2	1, 2, 1
3	1, 4, 6, 4, 1
4	1, 8, 26, 45, 45, 26, 8, 1
5	1, 16, 98, 327, 681, 944, 897, 588, 262, 76, 13, 1
6	1, 32, 342, 1943, 6982, 17326, 31236, 42198, 43521, 34601, 21249, 10020, 3571, 933, 169, 19, 1
7	1, 64, 1138, 10275, 58093, 228396, 664200, 1486921, 2633161, 3759650, 4386239, 4218971, 3363558, 2227042, 1222927, 554147, 205256, 61206, 14351, 2550, 323, 26, 1

Table: The first few f-vectors of CRY_n.

Section 2

Faces of Flow Polytopes

Faces of $\mathcal{F}_G(\mathbf{a})$ as subgraphs of G

- One particularly powerful use of flow polytopes arises from the following theorem:

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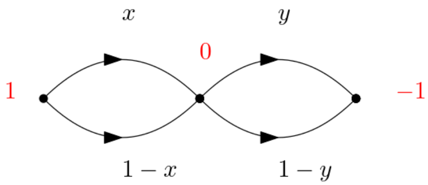
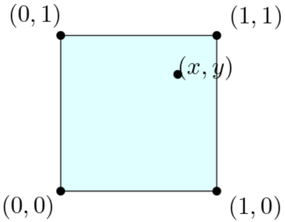
Theorem (Hille 2003; Gallo–Sodini 1978)

The d-dimensional faces of $\mathcal{F}_G(\mathbf{a})$ correspond to subgraphs H of G that have **1st Betti number** *$(|E|-|V|+c)$ equal to d and which are the support of an* **a***-valid flow, where c is the number of connected components of H.*

Example of Faces as Subgraphs





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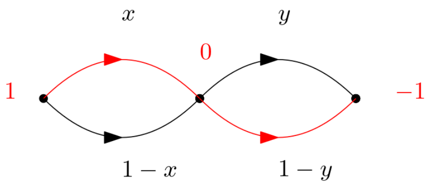

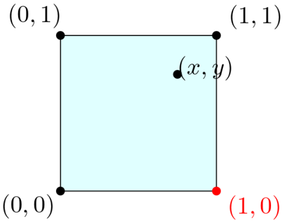

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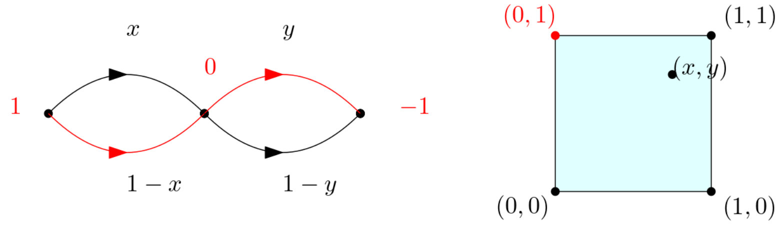
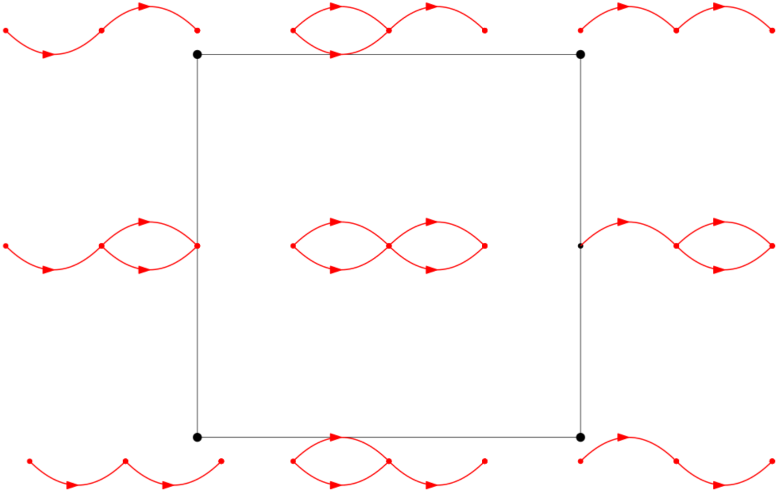


Example (cont.)



Example (cont.)

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William T. Dugan

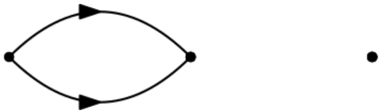
Flow Polytopes

Faces of Flow Polytopes

Main Results

Comments on Face Lattice and Final Remarks

- However, other subgraphs do not correspond to faces as they are not $(1, 0, -1)$-valid (i.e. their edges are not the support of some flow).



Section 3

Main Results

Andresen–Kjeldsen

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ON CERTAIN SUBGRAPHS OF A COMPLETE TRANSITIVELY DIRECTED GRAPH

E. ANDRESEN
Institute of Mathematics, University of Oslo, Oslo, Norway

K. KJELDSEN
Headquarters Defence Command Norway, Oslo, Norway

Received 3 June 1974

Let G_n be a complete transitively directed graph with $n + 1$ vertices $v_0, v_1, \ldots, v_n$. Let $\psi(n)$ be the number of subgraphs H of G_n where each vertex in H lies along a directed path from v_0 to v_n in H. $\psi(n)$ and some related quantities are calculated.

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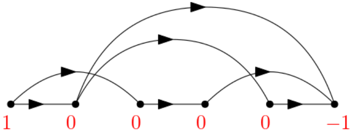

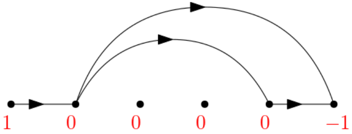

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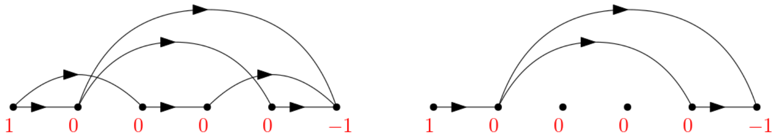
$$\Omega_n' := \{H \in \Omega_n \mid V(H) = \{v_1, \ldots, v_{n+1}\}, \text{ and single connected component}\}.$$



- This definition works for more general **a** as well!

Primitive f-vector

- From Hille's theorem, the f-polynomial of a flow polytope is a generating function over the set of $\mathbf{a}$-valid subgraphs which keeps track of first Betti number.

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Definition

The **primitive f-vector** of $\mathbf{Flow}_n(\mathbf{a})$, denoted $\widetilde{f}^{(n)}(\mathbf{a})$ (or as $\widetilde{f}^{(n)}(\mathbf{a}; x)$ if written as a polynomial) is a generating function over the set of $\mathbf{a}$-valid subgraphs of K_{n+1} that are primitive (use the entire vertex set) keeping track of the first Betti number.

Some more data

n	$\widetilde{f}$ of CRY_n
1	0, 1
2	0, 1, 1
3	0, 1, 4, 4, 1
4	0, 1, 11, 33, 42, 26, 8, 1
5	0, 1, 26, 171, 507, 840, 865, 584, 262, 76, 13, 1
6	0, 1, 57, 718, 4017, 12866, 26831, 39268, 42211, 34221, 21184, 10015, 3571, 933, 169, 19, 1
7	0, 1, 120, 2682, 25531, 138080, 490079, 1242533, 2375965, 3553184, 4258940, 4158866, 3342132, 2221444, 1221913, 554033, 205250, 61206, 14351, 2550, 323, 26, 1

Table: The first few primitive f-vectors of CRY_n.

f-vectors and primitive f-vectors

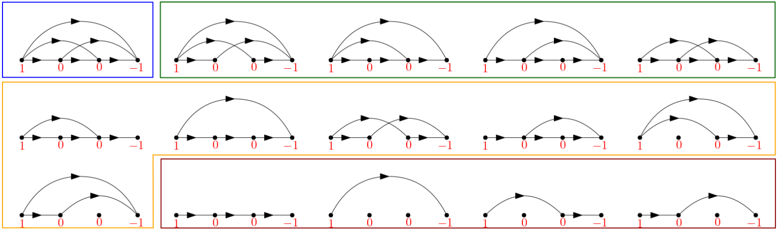

Figure: The elements of Ω_3 grouped by first Betti number, corresponding to the f-vector $(1, 4, 6, 4, 1)$ of CRY_3. The primitive f-vector is $(0, 1, 4, 4, 1)$.

Obtaining f-vector from primitive f-vector

Lemma (D. 2024)

For all $n \in \mathbb{N}$ and non-negative $\mathbf{a}$ *of length n:*

$$f^{(n)}(\mathbf{a}; x) = \frac{1}{x} + \sum_{\mathbf{b} \preceq \mathbf{a}} k_{\mathbf{a},\mathbf{b}} \widetilde{f}^{(\ell(\mathbf{b}))}(\mathbf{b}; x) \tag{1}$$

where $\mathbf{b} \preceq \mathbf{a}$ *if* $\mathbf{b}$ *can be obtained from* $\mathbf{a}$ *by deleting some subset of the zeros in* $\mathbf{a}$, $\ell(\mathbf{b})$ *is the length of* $\mathbf{b}$ *and where* $k_{\mathbf{a},\mathbf{b}}$ *is the number of ways of deleting* 0*'s from* $\mathbf{a}$ *to obtain* $\mathbf{b}$*..*

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$$\begin{aligned} f^{(6)}(1,0,0,1,1,0;x) &= \frac{1}{x} + \widetilde{f}^{(6)}(1,0,0,1,1,0;x) \\ &+ 2\widetilde{f}^{(5)}(1,0,1,1,0;x) + \widetilde{f}^{(5)}(1,0,0,1,1;x) \\ &+ \widetilde{f}^{(4)}(1,1,1,0;x) + 2\widetilde{f}^{(4)}(1,0,1,1;x) + \widetilde{f}^{(3)}(1,1,1;x) \end{aligned}$$

Proof of Lemma

- The set of graphs of all faces surjects on to the set of those that are primitive by restricting to the support of the netflow vector.

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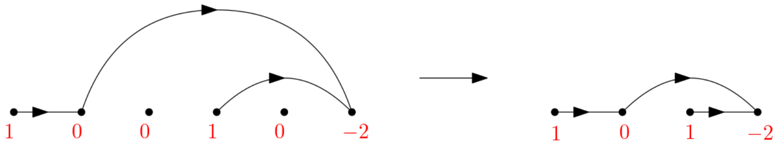

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Example

If $\mathbf{a} = (1, 1, 0, 0, 1, 0, 1, 0)$, we get blocks $(0, 1)$, $(0, 1)$, $(0, 0, 1)$, and (1). Hence $\text{revcomp}(\mathbf{a}) = (2, 2, 3, 1)$.

Connection to Quasisymmetric Polynomials

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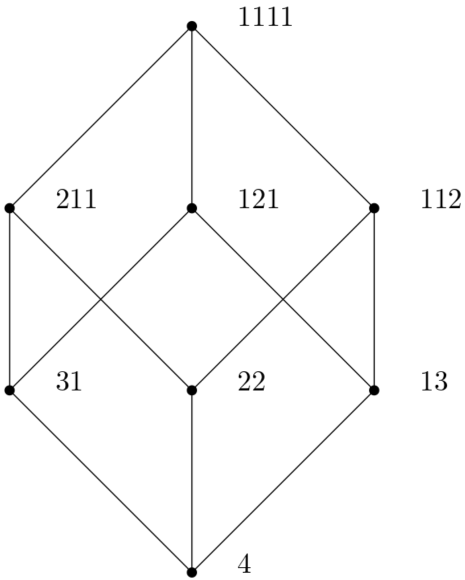
- Similar to the change-of-basis formula for writing a monomial quasisymmetric function in terms of Gessel's **fundamental quasisymmetric functions**.

Reverse Refinement Composition Poset



Formula for Primitive f-vector

Lemma (D. 2024)

For all $n \in \mathbb{N}$ and non-negative $\mathbf{a}$ of length n, let α be the composition of n given by $\alpha = \mathrm{revcomp}(\mathbf{a})$. Then the primitive f-vector of $\mathbf{Flow}_n(\mathbf{a})$ written as a polynomial is given by:

$$\widetilde{f}^{(n)}(\mathbf{a};x) = \frac{1}{x^n} P_\alpha(x, (x+1)^2 - 1, \ldots, (x+1)^n - 1) \tag{3}$$

Proof Sketch 1

- Associate to $T \subseteq \{v_2, \ldots v_n\}$ its indicator set $S_T \subseteq [n-1]$ in the canonical way (namely $i \in S_T$ if and only if $v_{i+1} \in T$).

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- In other words, R_S is the set of primitive subgraphs of K_{n+1} for which all vertices have nonzero out-degree and for which S keeps track of vertices (indices shifted by 1) which are *allowed to have* non-zero in-degree.

Example of R_S

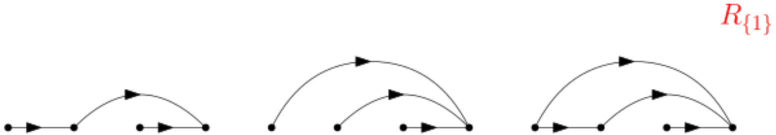

Figure: An example of R_S when $n = 3$ and $S = \{1\}$

Proof Sketch 2

- We can use inclusion-exclusion on these nested sets of graphs:

$$|Prim_{\mathbf{a}}| = \sum_{S \in [\mathrm{supp}(\mathbf{a}'), [n-1]]} (-1)^{|S|+n+1} |R_S|, \qquad (4)$$

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- Hence for each S, we can enumerate the number of graphs keeping track of their total out-degree.

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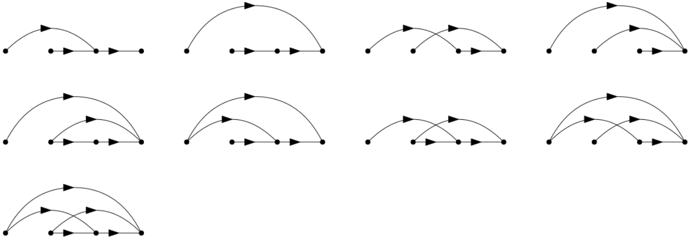

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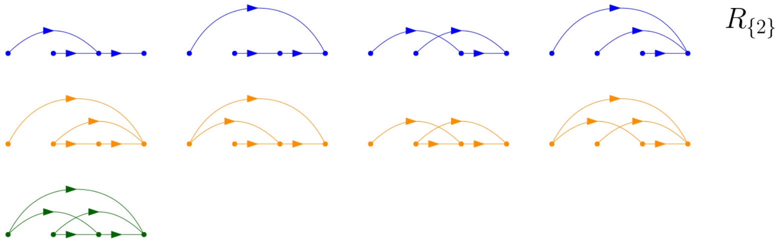

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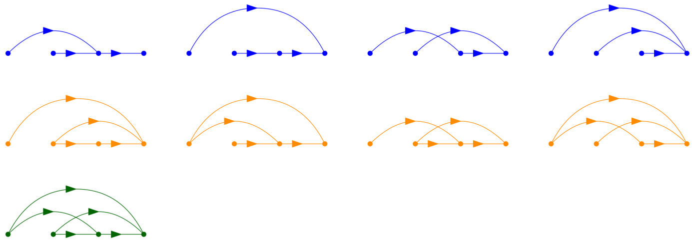

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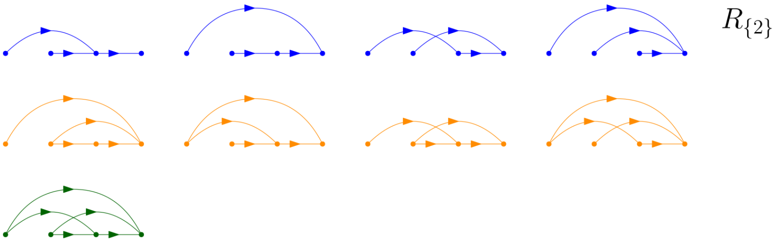

$$r_S(x) = x^5 + 4x^4 + 4x^3 = (x) \cdot ((x+1)^2 - 1) \cdot ((x+1)^2 - 1)$$

- In general, $r_S(x) = \prod_{x_i}^{\ell(\alpha_S)} (x+1)^{\alpha_i} - 1 = \mathbf{x}^{\alpha_S}|_{x_i = (x+1)^i - 1}$.

Proof Sketch 4

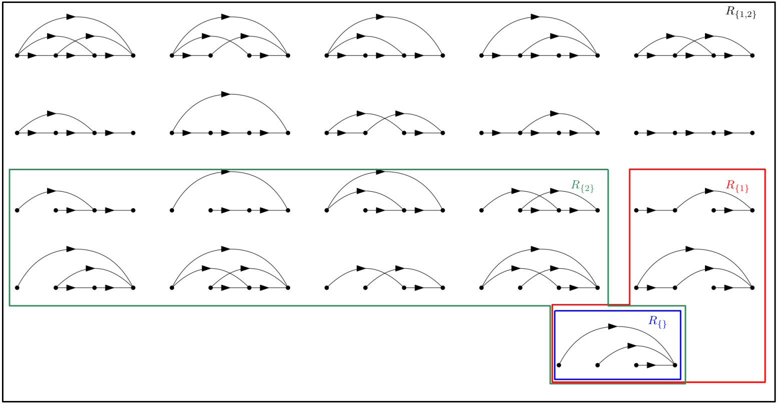

Figure: The sets of primitive graphs R_S for $n = 3$, as described in the proof of the Lemma.

Formula for Primitive f-vector (cont.)

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Corollary (D. 2024)

Let $\widetilde{f}^{(n)}(x)$ be the primitive f-vector of CRY_n written as a polynomial. Then for all $n \geq 1$:

$$\widetilde{f}^{(n)}(x) = \frac{1}{x^n} \sum_{m=0}^{n-1} (-1)^m \pi_{n-m}(x) \cdot h_m((x+1)^1 - 1, (x+1)^2 - 1, \ldots, (x+1)^{n-m} - 1)$$

where $\pi_n(x) := x^n [n]_{x+1}! = \prod_{i=1}^{n} ((x+1)^i - 1)$ and where h_k is a complete homogeneous symmetric polynomial.

Main Result

Theorem (D. 2024)

Given a netflow vector $(\mathbf{a}, -\sum_{i=1}^{n} a_i) = (a_1, \ldots a_n, -\sum_{i=1}^{n} a_i)$ *with* $a_i \in \mathbb{N}$, *let* α *be the integer composition of* n *given by* $\alpha = \text{revcomp}(\mathbf{a})$. *Then the* f*-vector of* $\mathbf{Flow}_n(\mathbf{a})$ *written as a Laurent polynomial is given by:*

$$f(\mathbf{a}; x) = \frac{1}{x} + \frac{1}{x^n} \sum_{\beta \succeq \alpha} (-1)^{\ell(\alpha)-\ell(\beta)} \pi_{\ell(\beta)}(x) \mathbf{x}^{\beta - \mathbf{1}} |_{x_i = (x+1)^i - (x+1)}$$

where $\pi_n(x) := x^n [n]_{x+1}! = \prod_{i=1}^{n} ((x+1)^i - 1)$.

f-vector of CRY_n

Corollary (D. 2024)

Let $f^{(n)}(x)$ be the f-vector of $CRY_n = \mathbf{Flow}_n(1, 0, \ldots, 0)$ written as a Laurent polynomial. Then for all $n \geq 1$:

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- We recover Andresen–Kjeldsen's result by setting $x=1$:

$$|\Omega_n| = \sum_{m=0}^{n-2}(-2)^m \pi_{n-m}\cdot h_m(2^1-1,2^2-1,\ldots,2^{n-m-1}-1)$$

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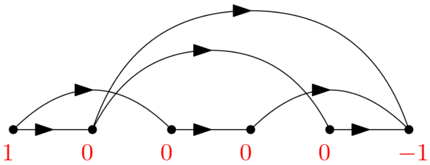
- We also recover $f(\mathbf{1};x) = [n]_{x+1}!$ as was proven for the Tesler polytope.

Primitive faces $\longleftrightarrow$ primitive Fishburn matrices

- A **Fishburn matrix** is an upper-triangular matrix with natural number entries such that no column nor row is the zero vector (**primitive** if using only entries in $\{0, 1\}$).
- There is a bijection between primitive faces of CRY and primitive Fishburn matrices:



$$\longleftrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure: The correspondence between elements of Ω'_n and primitive Fishburn matrices.

A generating function

Theorem (D. 2024)

The number of d-dimensional faces of CRY_n is given by the coefficient $f_d^{(n)} = [t^n x^d]F(t,x)$, where $F(t,x)$ is defined by:

$$F(t,x) := \frac{1}{x - xt} + \sum_{n=0}^{\infty} t^n x^{-n} \prod_{i=1}^{n} \frac{(1+x)^i - 1}{1 + ((1+x)^i - 1 - x)tx^{-1}}.$$

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- $F(t,x)$ may be obtained using the results above, or by a careful evaluation of a multivariate formula counting *Fishburn matrices* due to Jelínek (2011).

Section 4

Comments on Face Lattice and Final Remarks

Another relationship between $f(x)$ and $\widetilde{f}(x)$

Theorem (D. 2024)

For all $n \geq 1$, the f-vector and primitive f-vector of CRY_n are related as:

$$x \cdot f^{(n)}(x) = (1+x)^{n-1}\widetilde{f}^{(n)}(x). \tag{5}$$

where the **primitive f-vector**, $\widetilde{f}^{(n)}(x)$, *enumerates subgraphs having all vertices $v_1, \ldots, v_{n+1}$ in a single connected component.*

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- We believe this formula may lead to an understanding of the face lattice of CRY_n, which is current work in progress.
- We are also now working to extend our results to various families of subgraphs of K_{n+1}.

Open Problems

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1. Find a combinatorial proof of the previous theorem.
2. Does the expression as a (quasi-)symmetric polynomial *before* evaluating have some deeper meaning/correspond to some other combinatorial object? If so, what is this object? What do other evaluations mean?
3. Use the techniques in this talk (especially primitive f-vectors) to compute the f-vectors of other flow polytopes.

Selected References

Andresen, E. and Kjeldsen, K. (1976).
On certain subgraphs of a complete transitively directed graph.
Discrete Mathematics, 14(2):103–119.

Dugan, W. T. (2024).
On the f-vectors of flow polytopes for the complete graph.
arXiv preprint arXiv:2409.15519.

Gallo, G. and Sodini, C. (1978).
Extreme points and adjacency relationship in the flow polytope.
Calcolo, 15(3):277–288.

Hille, L. (2003).
Quivers, cones and polytopes.
Linear Algebra and its Applications, 365:215–237.

Thank You!