Colored Schur-like bases and generalizations of the symmetric functions

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Outline

Introduction

- *Sym*, *QSym*, and *NSym*
- Schur-like bases
- $NSym_A$ and $QSym_A$

Colored generalizations of Schur-like bases

- Colored tableaux and bases of $QSym_A$
- Properties of Schur-like bases of $NSym_A$

Colored generalizations of Sym

- $PSym_A$ and Sym_A
- Colored Schur and dual Schur functions

[1] A partially commutative generalization of dual immaculate functions. D. 2024. Electron. J. Combin.
[2] Schur-like bases and their colored generalizations. D. 2024. Doctoral Thesis.

INTRODUCTION

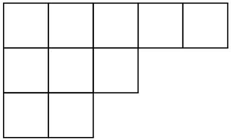
The Basics

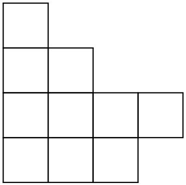
Partition: $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$ if $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell$ and $\sum \lambda_i = n$.

Composition: $\alpha = (\alpha_1, \ldots, \alpha_\ell) \models n$ if $\sum \alpha_i = n$.

Example

$(5, 3, 2)$ is a partition of 10, and $(1, 2, 4, 3)$ is a composition of 10.





The Symmetric Functions

A *symmetric function* $f(x)$ is a formal power series such that

$$f(x_{\omega(1)}, x_{\omega(2),\dots}) = f(x_1, x_2, \dots)$$

for any permutation ω of $\mathbb{N}$.

Example

$f = x_1x_2^2 + x_1x_3^2 + \dots + x_2x_1^2 + x_2x_3^2 + \dots + x_3x_1^2 + x_3x_2^2 + x_3x_4^2 + \dots$

Complete homogeneous basis:

$$h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} \qquad h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$$

The Quasisymmetric Functions

A *quasisymmetric function* $f(x)$ is a formal power series where the coefficients of monomials $x_{i_1}^{a_1} \dots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} \dots x_{j_k}^{a_k}$ are equal if $i_1 < \dots < i_k$ and $j_1 < \dots < j_k$.

Example

$f = x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 + \dots x_2 x_3^2 + x_2 x_4^2 + \dots x_3 x_4^2 + \dots$

The ring of quasisymmetric functions is written $QSym$.

The monomial basis:

$$M_\alpha = \sum_{i_1 < \dots < i_k} x_{i_1}^{\alpha_1} \cdots , x_{i_k}^{\alpha_k}$$

The Non-commutative Symmetric Functions

The ring of non-commutative symmetric functions, written *NSym*, is defined on noncommutative generators $H_1, H_2, \ldots$ as

$$NSym = \mathbb{Q} \langle H_1, H_2, \cdots \rangle .$$

For a composition α, the complete homogeneous non-commutative symmetric function is defined $H_\alpha = H_{\alpha_1} \cdots H_{\alpha_{\ell(\alpha)}}$.

Definition

The forgetful map $\chi : NSym \to Sym$ maps complete homogeneous non-commutative symmetric functions to complete homogeneous symmetric functions

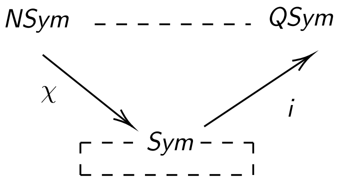
$$\chi(H_\alpha) = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}} = h_{sort(\alpha)} \in Sym.$$

The Symmetric Functions and their Generalizations

Symmetric functions are generalized by **noncommutative symmetric functions** (NSym) and **quasisymmetric functions** (QSym). All three admit a **Hopf algebra** structure.

$$Sym = \mathbb{Q}[h_n : n \in \mathbb{N}] \quad \text{where} \quad h_m h_n = h_n h_m$$

$$NSym = \mathbb{Q}\langle H_n : n \in \mathbb{N}\rangle \quad \text{where} \quad H_m H_n \neq H_n H_m$$



bases indexed by compositions

bases indexed by partitions

The forgetful map $\chi : NSym \to Sym$ is defined as $\chi(H_n) = h_n$.

The map $i : Sym \to QSym$ is the inclusion of Sym into $QSym$.

The Schur Functions

Definition

A **semistandard Young tableau** (SSYT) of shape $\lambda \vdash n$ is a filling of the diagram of λ with positive integers such that the numbers are weakly increasing in rows and strictly increasing in columns.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 4 \\ \hline \end{array}$$

The Schur Functions

The monomial x^T corresponding with the SSYT to the right is $x_1 x_2^2 x_3$.

$$\mathrm{T} = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$$

Definition

The **Schur symmetric functions** are defined as

$$s_\lambda = \sum_T x^T,$$

where the sum runs over all semistandard Young tableaux T of shape λ with entries in $\mathbb{Z}_{>0}$.

$$s_{(2,2)} = x_1^2 x_2^2 + x_1^2 x_2 x_3 + x_1^2 x_3^2 + x_1 x_2^2 x_3 + \cdots$$

Schur-like Bases

A basis $\{\mathcal{S}_\alpha\}_\alpha$ of *NSym* is **Schur-like** if $\chi(\mathcal{S}_\lambda) = s_\lambda$.

The dual basis $\{\mathcal{S}^*_\alpha\}_\alpha$ in *QSym* is also considered Schur-like, and is usually defined via tableaux that generalize SSYT.

Primary Schur-like Bases

Immaculate $\{\mathfrak{S}_\alpha\}_\alpha$ **and dual immaculate** $\{\mathfrak{S}^*_\alpha\}_\alpha$

Shin $\{\text{ש}_\alpha\}_\alpha$ **and extended Schur** $\{\text{ש}^*_\alpha\}_\alpha$

Young noncommutative Schur $\{\hat{\mathbf{s}}_\alpha\}_\alpha$ **and Young quasisymmetric Schur** $\{\hat{\mathbf{s}}^*_\alpha\}$

Schur-like Bases in QSym

Dual immaculate:
weakly increasing rows, strictly increasing first column

Extended Schur (dual shin):
weakly increasing rows, strictly increasing columns

Young quasisymmetric Schur:
weakly increasing rows, strictly increasing first column, triple rule

1	3	4	
2	2	5	6

Immaculate

1	1	3	
2	3	4	5

Shin

1	1	4	
2	3	3	5

SSYCT

Properties of Schur-like Bases in *QSym*

Dual immaculate: If $\ell(\lambda) = k$,

$$s_\lambda = \sum_{\sigma \in S_k} (-1)^\sigma \mathfrak{S}^*_{\lambda_{\sigma_1}+1-\sigma_1, \ldots, \lambda_{\sigma_k}+k-\sigma_k}$$

where the sum runs over σ such that $\lambda_{\sigma_i} + i - \sigma_i > 0$.

Young quasisymmetric Schur: $s_\lambda = \sum_{sort(\alpha)=\lambda} \hat{\mathbf{s}}^*_\alpha$

Extended Schur: $s_\lambda = \mathfrak{w}^*_\lambda$

Colored Variables

Let $A = \{a_1, \ldots a_m\}$ be a finite alphabet of *colors*.

Word: a finite string of colors from A

Sentence: a finite sequence of words

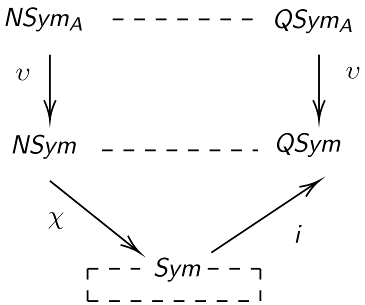
Example

Let $A = \{a, b, c\}$. The sentence $(aba, cc, bcab)$ is composed of the words *aba*, *cc*, and *bcab*.

If A contains only one color, sentences are in bijection with compositions. For example, $(aaa, a, aa) \leftrightarrow (3, 1, 2)$.

Consider partially commutative variables $x_{a_i, j}$ for $a_i \in A$ and $j \in \mathbb{N}$.

The Symmetric Functions and their Generalizations



bases indexed by sentences

bases indexed by compositions

bases indexed by partitions

$$Sym = \mathbb{Q}[h_n : n \in \mathbb{N}] \quad \text{where} \quad h_m h_n = h_n h_m$$

$$NSym = \mathbb{Q}\langle H_n : n \in \mathbb{N}\rangle \quad \text{where} \quad H_m H_n \neq H_n H_m$$

$$NSym_A = \mathbb{Q}\langle H_w : \text{ words } w\rangle \quad \text{where} \quad H_w H_v \neq H_v H_w$$

COLORED GENERALIZATIONS OF SCHUR-LIKE BASES

Colored Tableaux

Colored diagrams:

$$(ab, bca) \longrightarrow$$

a	b	
b	c	a

Definition

Colored tableaux of shape l are colored diagrams of l filled with positive integers.

$a, 1$	$b, 1$	
$b, 2$	$c, 2$	$a, 2$

$a, 1$	$b, 2$	
$b, 2$	$c, 2$	$a, 2$

$a, 1$	$b, 3$	
$b, 2$	$c, 2$	$a, 4$

$a, 1$	$b, 4$	
$b, 2$	$c, 2$	$a, 3$

Colored Tableaux

Definition

The **type** of a colored tableau is a sentence $C = (u_1, ..., u_g)$ where u_i lists the colors in boxes filled with i's in a particular order.

For a colored tableau T of type $(u_1, u_2, \cdots, u_g)$, we say that

$$x_T = x_{u_1,1} x_{u_2,2} \cdots x_{u_g,g}$$

Example

The type of this colored immaculate tableau is (a, cb, b) which corresponds to $x_{a,1} x_{cb,2} x_{b,3}$

$a, 1$	$b, 2$
$c, 2$	$b, 3$

Colored Schur-like Functions in $QSym_A$

Definition

The **colored dual immaculate functions**, the **colored extended Schur functions**, and the **colored Young quasisymmetric Schur functions** are defined by

$$\mathfrak{S}^*_J = \sum_T x_T, \qquad \boldsymbol{\mathfrak{w}}^*_J = \sum_T x_T, \qquad \text{and} \qquad \hat{\mathbf{s}}^*_J = \sum_T x_T,$$

where the sums run over colored immaculate tableaux T of shape J, colored shin tableaux T of shape J, and colored Young composition tableaux T of shape J, respectively.

Results on Schur-like Functions in $QSym_A$

Other results on the colored Schur-like bases in $QSym_A$ include:

- Expansions into the colored monomial basis
- Expansions into the colored fundamental basis
- Definition of skew functions
- Posets, inverse expansions, and comultiplication

Colored Schur-like Bases in $NSym_A$

Each of the three colored Schur-like bases in $NSym_A$ is defined differently, either using creation operators, a Pieri rule, or duality.

Results on these bases include:

- Expansions to and from bases
- Pieri rules
- Structure coefficients
- Multiplicative properties

Colored Immaculate Creation Operators

Definition

The **colored non-commutative Bernstein operator** $\mathbb{B}_w$ is defined as

$$\mathbb{B}_w = \sum_u \sum_Q (-1)^i H_{w \cdot u}\left(\sum_{Q \preceq S} M_S^{\perp}\right),$$

where the sum runs over all $Q = (q_1, ..., q_i)$ such that $q_i \cdots q_2 q_1 = u$.

Example

$\mathbb{B}_{abc}(H_{def}) = H_{abc,def} - H_{abcf,de} + H_{abcfe,d} - H_{abcef,d} - H_{abcdef} + H_{abcefd} + H_{abcfde} - H_{abcfed}$

Colored Immaculate Functions

Definition

For sentence $J = (v_1, ..., v_h)$, we define the **colored immaculate function** as

$$\mathfrak{S}_J = \mathbb{B}_{v_1}\mathbb{B}_{v_2}...\mathbb{B}_{v_h}(1).$$

Equivalently, we have

$$\mathfrak{S}_{(v_1,v_2,...,v_h)} = \mathbb{B}_{v_1}(\mathfrak{S}_{(v_2,...,v_h)}).$$

Example

$\mathfrak{S}_{(abc,def)} = \mathbb{B}_{abc}(\mathfrak{S}_{def}) = \mathbb{B}_{abc}(H_{def}) = H_{abc,def} - H_{abcf,de} + H_{abcfe,d} - H_{abcef,d} - H_{abcdef} + H_{abcefd} + H_{abcfde} - H_{abcfed}$

The Colored Immaculate Descent Graph

Definition

The **colored immaculate descent graph** $\mathfrak{D}_A^n$ is the directed graph whose vertices are sentences of size n in alphabet A and there is an edge from I to J (weighted with $L_{I,J}$) if there exists a standard colored immaculate tableau of shape I with colored descent composition J.

Example

$a,1$	$b,4$	
$c,2$	$b,3$	$b,5$

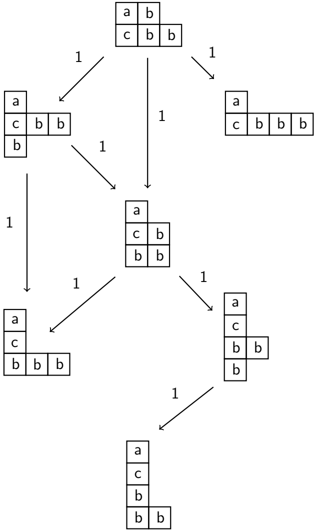
(a,cbb,b)

$a,1$	$b,3$	
$c,2$	$b,4$	$b,5$

(a,cb,bb)

$a,1$	$b,5$	
$c,2$	$b,3$	$b,4$

(a,cbbb)



The Colored Immaculate Descent Graph

Corollary [D. 2023]

For a sentence J,

$$\mathfrak{S}_J = \sum_I b_{I,J} R_I,$$

where the sum runs over all sentences I and

$$b_{I,J} = \sum_{\mathcal{P}} (-1)^{k-1} L_{(J_1,J_2)} L_{(J_2,J_3)} \cdots L_{(J_{k-1},J_k)},$$

where the sum runs over all paths $\mathcal{P} = \{J = J_k \leftarrow J_{k-1} \leftarrow \ldots \leftarrow J_2 \leftarrow J_1 = I\}$ from I to J in $\mathfrak{D}_A^{|J|}$.

Corollary [D. 2023]

For a composition $\beta \models n$ and a partition $\lambda \vdash n$,

$$\mathfrak{S}_\beta = \sum_{\alpha \models n} L^{-1}_{\alpha,\beta} R_\alpha \qquad \text{with} \qquad L^{-1}_{\alpha,\beta} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P}-1)} prod(\mathcal{P}),$$

where the sum runs over directed paths $\mathcal{P}$ from α to β in $\mathfrak{D}^n$, and

$$s_\lambda = \sum_{\alpha \models n} L^{-1}_{\alpha,\lambda} r_\alpha \quad \text{with} \quad L^{-1}_{\alpha,\lambda} = \sum_{\mathcal{P}} (-1)^{\ell(\mathcal{P}-1)} prod(\mathcal{P}),$$

where the sum runs over directed paths $\mathcal{P}$ from α to λ in $\mathfrak{D}^n$.

Colored Shin Functions

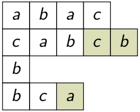
We write $J \subset^{\shin}_{u} I$ if I can be obtained by adding boxes to J such that we do not create a row in I such that the original row in J is shorter than any row in I below it. These added boxes must form the word u when read from left to right, bottom to top.

Definition

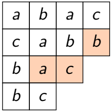
The **shin functions** $\{\shin_I\}_I$ are the unique set of functions satisfying

$$\shin_I H_w = \sum_{J \subset^{\shin}_{w} I} \shin_J .$$

I = (abac, cab, b, bc)
w = acb



example



non-example

Colored Noncommutative Schur Functions

The **colored Young noncommutative Schur functions** are the duals to the colored Young quasisymmetric Schur functions.

Theorem [D. 2024]

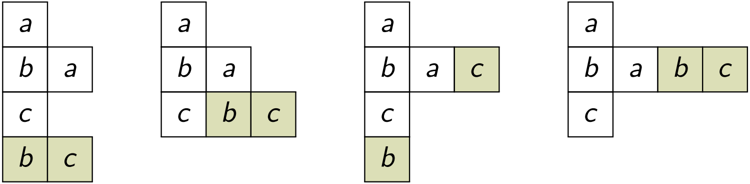
Let I be a sentence and w a word. Then,

$$\check{\mathbf{s}}_I H_w = \sum_J \check{\mathbf{s}}_J,$$

where the sum runs over J obtained by adding boxes to the right side of I such that no two boxes are added in the same column and a box can only be added to a row if there is no lower row of the same length. The colors in the new boxes should form the word w when read from left to right, bottom to top.

Colored Noncommutative Schur Functions

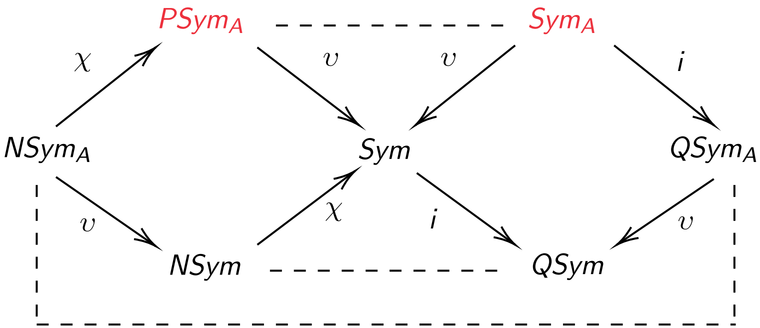
$$\hat{\mathbf{s}}_{(a,ba,c)}H_{(bc)} = \hat{\mathbf{s}}_{(a,ba,c,bc)} + \hat{\mathbf{s}}_{(a,ba,cbc)} + \hat{\mathbf{s}}_{(a,bac,c,b)} + \hat{\mathbf{s}}_{(a,babc,b)}.$$



COLORED GENERALIZATIONS OF THE SYMMETRIC FUNCTIONS

Relationships Between Algebras

We introduce two new algebras that complete this picture:



Sorted Sentences

A sentence is called a **p-sentence** if its words are sorted first in decreasing order by length and second lexicographically. Given a sentence I, we write $sort(I)$ for the unique associated p-sentence.

Example

$$P = (aba, bcc, ab, a, c)$$

$$P = sort(c, bcc, a, ab, aba) = sort(ab, bcc, a, aba, c) = \cdots$$

P-sentences are analogous to partitions as sentences are analogous to compositions. When A is an alphabet of size one, p-sentences are in bijection with partitions: $(aaa, aaa, aa) \leftrightarrow (3, 3, 2)$.

Colored Generalizations of Sym: $PSym_A$

Definition

For an alphabet A, the **algebra of p-sentences** is defined

$$PSym_A = \mathbb{Q}[h_w : \text{words } w].$$

Compared to Sym, $NSym$, and $NSym_A$:

$$Sym = \mathbb{Q}[h_n : n \in \mathbb{N}] \qquad PSym_A = \mathbb{Q}[h_w : \text{words } w]$$

$$NSym = \mathbb{Q}\langle H_n : n \in \mathbb{N}\rangle \qquad NSym_A = \mathbb{Q}\langle H_w : \text{ words } w\rangle$$

$NSym_A$ maps to $PSym_A$ by the **colored forgetful map:**

$$\chi : NSym_A \to PSym_A \qquad \text{defined} \qquad \chi(H_I) = h_{sort(I)}.$$

Colored Generalizations of *Sym*: Sym_A

Definition

Let Sym_A denote the set of **colored symmetric functions** $f \in \mathbb{Q}[x_A]$ such that for any permutation σ of $\mathbb{N}$,

$$f(x_{A,1}, x_{A,2}, \ldots) = f(x_{A,\sigma(1)}, x_{A,\sigma(2)}, \ldots).$$

In other words, if two monomials are associated with the same p-sentence, then they have the same coefficients.

Example

$$f = x_{a,1}x_{bc,2} + x_{bc,1}x_{a,2} + \cdots + x_{a,5}x_{bc,7} + x_{bc,5}x_{a,7} + \cdots \in Sym_A$$

$$g = x_{a,1}x_{bc,2} + 3x_{bc,1}x_{a,2} + \cdots \notin Sym_A$$

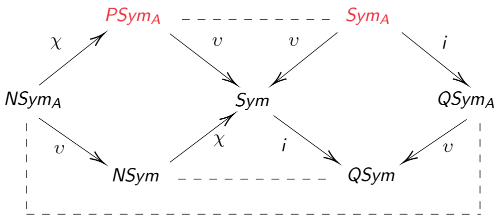
$PSym_A$ and Sym_A

Theorem [D. 2024]

$PSym_A$ and Sym_A are dual Hopf algebras.

Proposition [D. 2024]

If A is an alphabet of size one, then $PSym_A$ and Sym_A are both isomorphic to Sym.



Colored Schur and Dual Schur Functions

For a p-sentence P, the **colored dual Schur function** is defined as

$$s^*_P = \sum_Q \mathcal{K}_{P,Q} m_Q.$$

where $\mathcal{K}_{P,Q}$ denotes the number of colored semistandard Young tableaux of shape P and type Q.

$$s^*_{(abb,ca)} = m_{(abb,ca)} + m_{(ab,cb,a)} + m_{(ab,ca,b)}$$

$a,1$	$b,1$	$b,1$
$c,2$	$a,2$	

$a,1$	$b,1$	$b,2$
$c,2$	$a,3$	

$a,1$	$b,1$	$b,3$
$c,2$	$a,2$	

The **colored Schur functions** $\{s_P\}_P$ are defined as the duals in $PSym_A$ to the colored dual Schur functions in Sym_A.

Future Work

- Which properties of the Schur functions do the colored Schur and colored dual Schur functions generalize?
- How do the colored Schur and colored dual Schur bases relate to the various colored Schur-like bases of $NSym_A$ and $QSym_A$?
- What are the commutative images of the colored Schur-like bases? Do any subsets of these images form bases of $PSym_A$ that generalize the Schur functions?

THANK YOU!

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