Permutations without non-reduced co-BPDs

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Michigan State University

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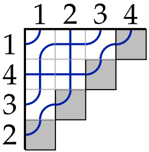
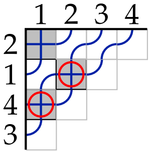
Pipe dreams

A *pipe dream* is a filling of $(n-1, \ldots, 2, 1)$ staircase using

 and 

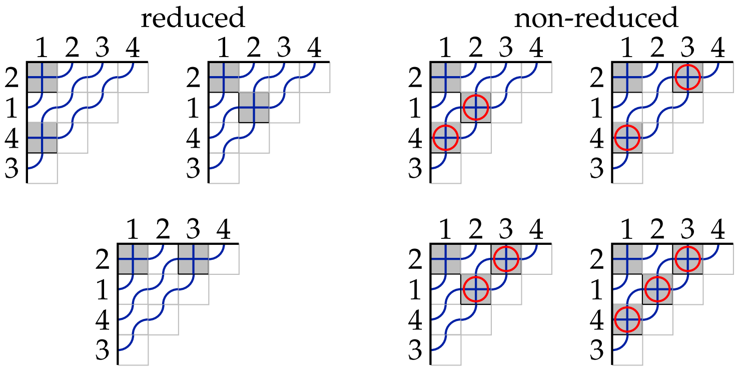
so that pipes enter from the top and exit out of the left.





Billey–Bergeron (1993) // Fomin–Kirillov (1994) // Knutson–Miller (2004)

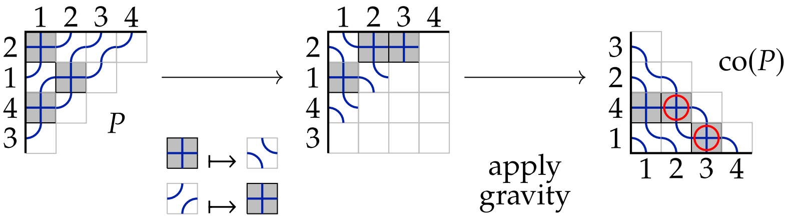
PD(2143)



$$\underbrace{\mathfrak{G}_{2143}}_{\text{Grothendieck}} = \underbrace{x_1x_3 + x_1x_2 + x_1^2}_{\text{Schubert } \mathfrak{S}_{2143}} - (x_1x_2x_3 + x_1^2x_3 + x_1^2x_2) + \underbrace{x_1^2x_2x_3}_{\text{Castelnuovo–Mumford}}$$

co-PD

For P in $\mathrm{PD}(w)$, corresponding co-PD is



"Traces" out the permutation v on the left read top-to-bottom

Grothendieck into Schubert

Theorem (Lenart, 1999)

Let $a_{w,v} = \#\{P \in \mathrm{PD}(w) : \mathrm{co}(P)$ *reduced & traces out* $v\}$. *Then*

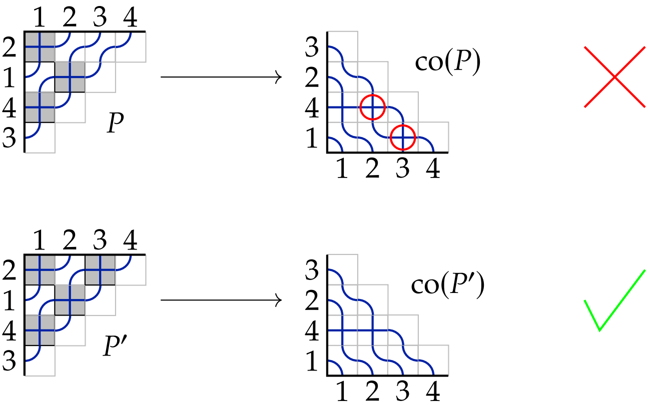
$$\mathfrak{G}_w = \sum_v (-1)^{\ell(v)-\ell(w)} a_{w,v} \cdot \mathfrak{S}_v.$$

Example

$$\mathfrak{G}_{2143} = \mathfrak{S}_{2143} - (\mathfrak{S}_{2341} + \mathfrak{S}_{3142}) + \mathfrak{S}_{3241}$$

$a_{2143,3241} = 1$

$$a_{2143,3241} = \#\{P \in \mathrm{PD}(2143) : \mathrm{co}(P) \text{ reduced \& traces } 3241\}$$



Schubert into Grothendieck

Write $\mathrm{pd}(w) \subseteq \mathrm{PD}(w)$ for the *reduced* PDs of w.

Theorem (Lascoux, 2004)

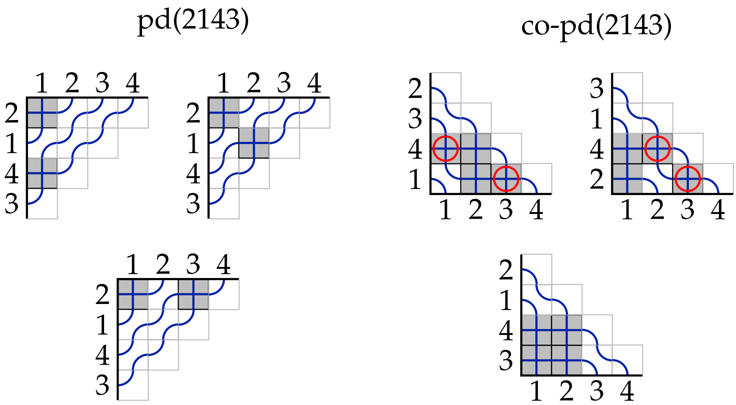
Let $b_{w,v} = \#\{P \in \mathrm{pd}(w) : \mathrm{co}(P)$ *traces out* $v\}$. *Then*

$$\mathfrak{S}_w = \sum_v b_{w,v} \cdot \mathfrak{G}_v.$$

Example

$$\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{2341} + \mathfrak{G}_{3142}$$

$b_{2143,v} = \#\{P \in \mathrm{pd}(2143) : \mathrm{co}(P) \text{ traces out } v\}$

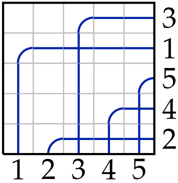


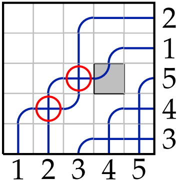
$$\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{2341} + \mathfrak{G}_{3142}$$

Bumpless pipe dreams

A *marked bumpless pipe dream* is a $n \times n$ filling using the tiles

so that pipes enter from the bottom and exit out of the right.

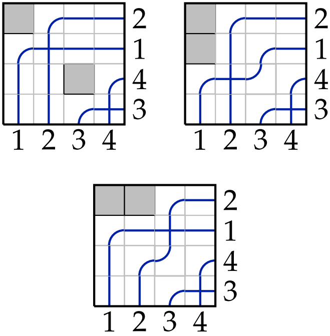




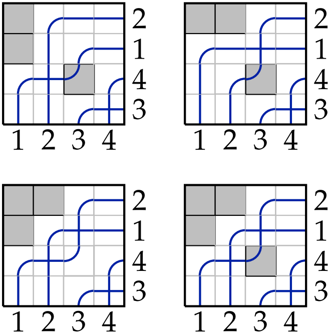
Lam–Lee–Shimozono (2018) // Weigandt (2021)

MBPD(2143)

reduced



non-reduced

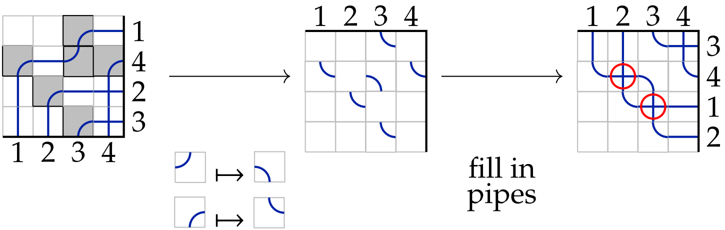


$$\mathfrak{G}_{2143} = x_1x_3 + x_1x_2 + x_1^2 \; - \; (x_1x_2x_3 + x_1^2x_3 + x_1^2x_2) \; + \; x_1^2x_2x_3$$

co-BPD

Only care about BPD(w) ⊆ MBPD(w) with no  tiles

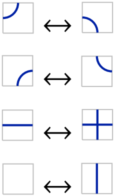
For B in BPD(w), corresponding co-BPD is

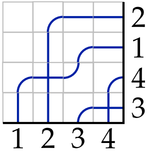


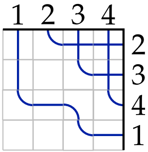
"Traces" out the permutation v on the right read top-to-bottom

co-BPD

Equivalently $B \leftrightarrow \text{co}(B)$ via the mapping







Grothendieck into Schubert (revisited)

Theorem (Weigandt, 2024+)

Let $a_{w,v} = \#\{B \in \mathrm{BPD}(w) : \mathrm{co}(B)$ *reduced & traces out* $v\}$. *Then*

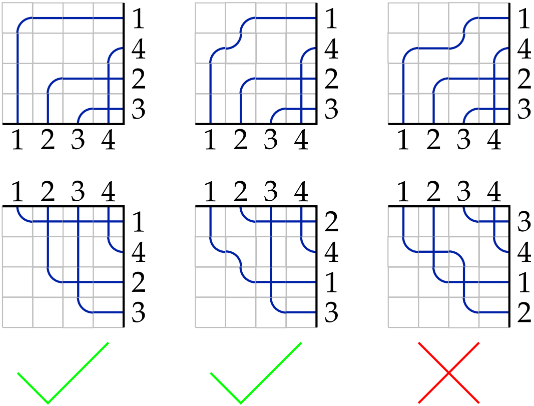
$$\mathfrak{G}_w = \sum_v (-1)^{\ell(v)-\ell(w)} a_{w,v} \cdot \mathfrak{S}_v.$$

Example

$$\mathfrak{G}_{1423} = \mathfrak{S}_{1423} - \mathfrak{S}_{2413}$$

$a_{1423,v} = \#\{B \in \mathrm{BPD}(1423) : \mathrm{co}(B) \text{ reduced \& traces } v\}$

$$\mathfrak{G}_{1423} = \mathfrak{S}_{1423} - \mathfrak{S}_{2413}$$



Schubert into Grothendieck

Write $\mathrm{bpd}(w) \subseteq \mathrm{BPD}(w)$ for the *reduced* BPDs of w.

Theorem (Weigandt, 2024+)

Let $b_{w,v} = \#\{B \in \mathrm{bpd}(w) : \mathrm{co}(B)$ *traces out* $v\}$. *Then*

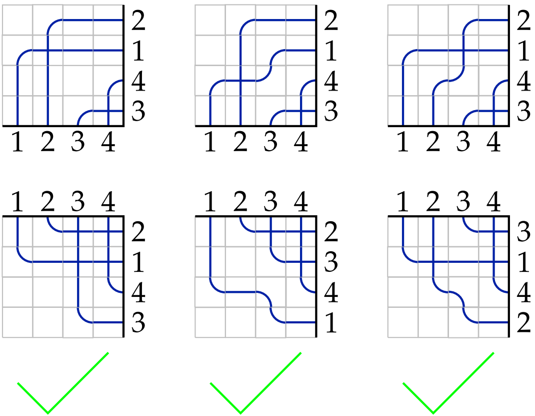
$$\mathfrak{S}_w = \sum_v b_{w,v} \cdot \mathfrak{G}_v.$$

Example

$$\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{2341} + \mathfrak{G}_{3142}$$

$b_{2143,v} = \#\{B \in \mathrm{bpd}(2143) : \mathrm{co}(B) \text{ traces } v\}$

$$\mathfrak{S}_{2143} = \mathfrak{G}_{2143} + \mathfrak{G}_{2341} + \mathfrak{G}_{3142}$$



Question

Open problem (Weigandt 2024)

Characterize w for which $\mathrm{co}(B)$ is *reduced* for every $B \in \mathrm{BPD}(w)$.

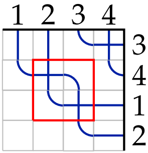
For pipe dreams it's dominant (i.e., 132-avoiding) permutations

Answer through S_9

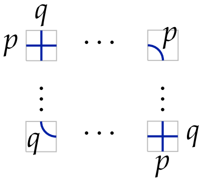
w has this property if and only if w avoids the seven patterns 1423, 12543, 13254, 25143, 215643, 216543, and 241653.

Observation

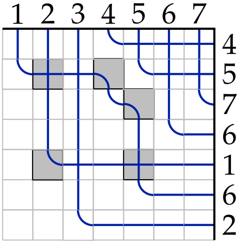
What does a double-crossing in a co-BPD look like?



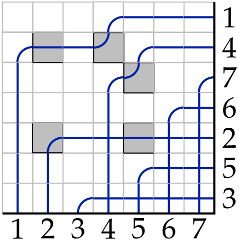
More generally,



Example



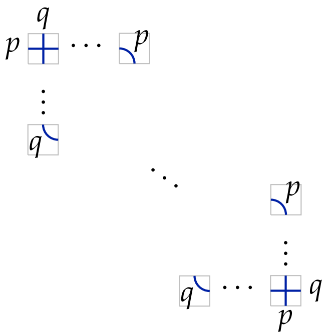
co(B)



B

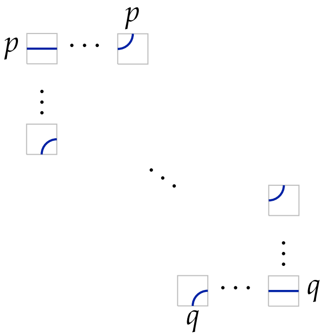
Observation

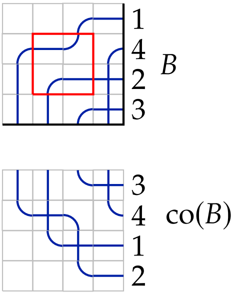
Actually, a co-BPD has a double-crossing if and only if it has:



Observation

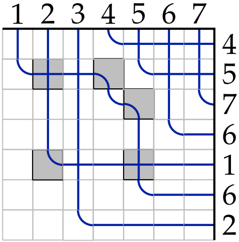
If a co-BPD is non-reduced, its corresponding BPD has:



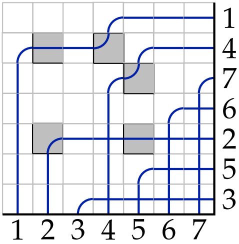


Call this *"the configuration"*

Example



co(B)



B

Outline

Property: "co(B) reduced for every $B \in \text{BPD}(w)$"

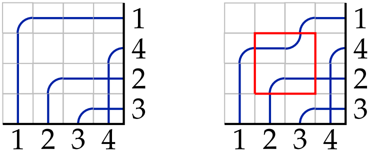
$$w \text{ has property} \iff \text{none of its BPDs have the config}$$
$$\overset{(??)}{\iff} w \text{ avoids those seven patterns}$$

(Nec.) if w contains a pattern then it has a BPD with config

(Suff.) if $B \in \text{BPD}(w)$ has config then w contains a pattern

Necessary to avoid

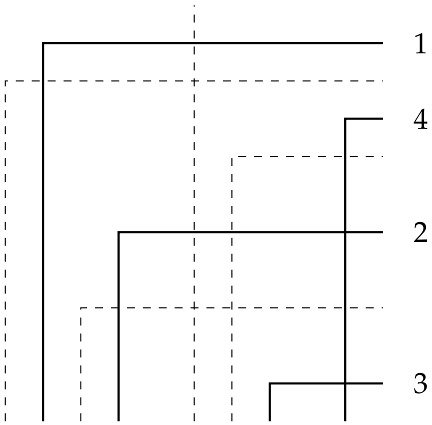
Rothe BPD is unique BPD with no  tiles



Perform “droop moves” from Rothe to get a BPD with config

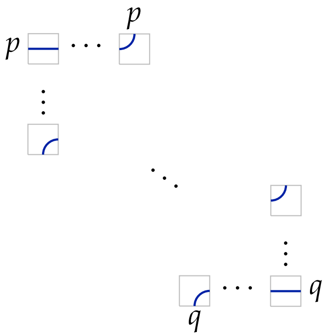
Necessary to avoid

Perform "droop moves" from Rothe to get a BPD with config



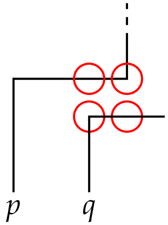
Sufficient to avoid

Suppose $B \in \mathrm{BPD}(w)$ has config

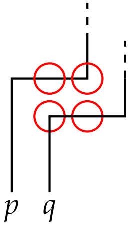


Sufficient to avoid

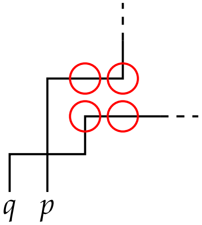
We consider three cases:



don't cross before &
q never droops



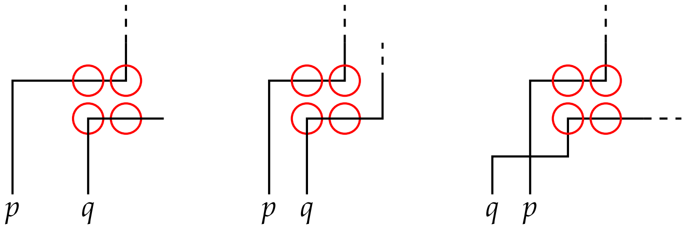
don't cross before &
q does droop



cross before

Sufficient to avoid

We consider three cases:



1423
13254

12543
215643

25143, 216543
241653

Thank you for listening!