Invariant Theory for the Face Algebra of the Braid Arrangement

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MSU Combinatorics + Graph Theory Seminar

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Outline

I. Faces of hyperplane arrangements

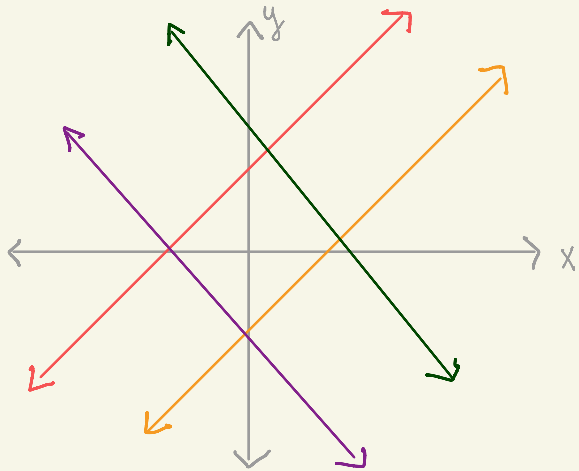
II. The Face Algebra + its history

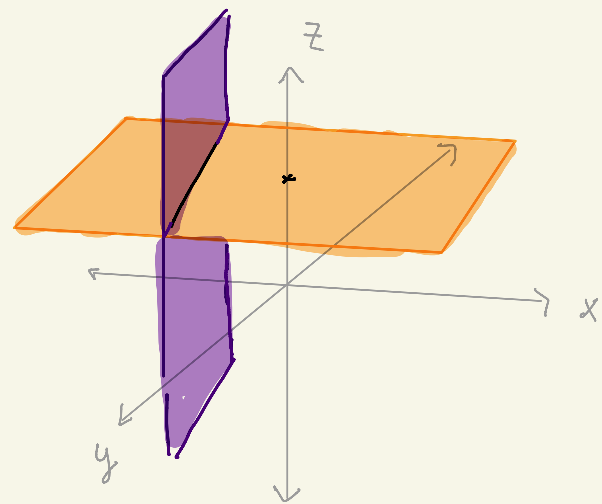
III. Connections to Solomon's descent algebra

IV. Invariant theory for the face algebra

Hyperplane Arrangements

A (real) hyperplane arrangement is a (finite) collection of $(n-1)$-dimensional affine subspaces of $\mathbb{R}^n$.

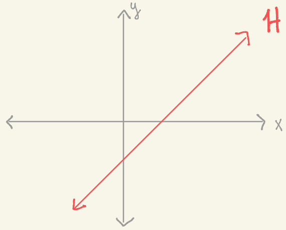


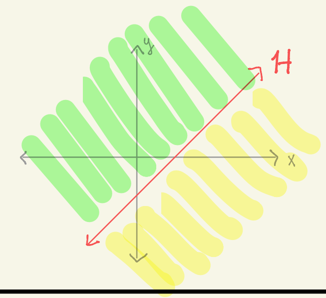


Faces of Hyperplane Arrangements

Each hyperplane partitions $\mathbb{R}^n$ into three subsets:

The hyperplane itself and Two half spaces





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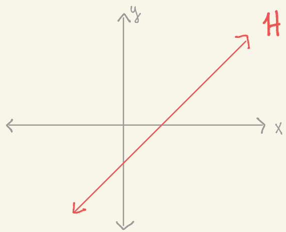
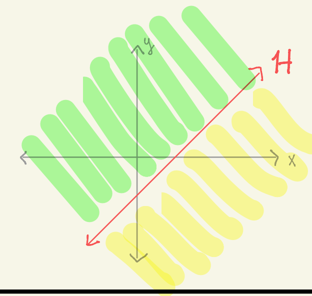
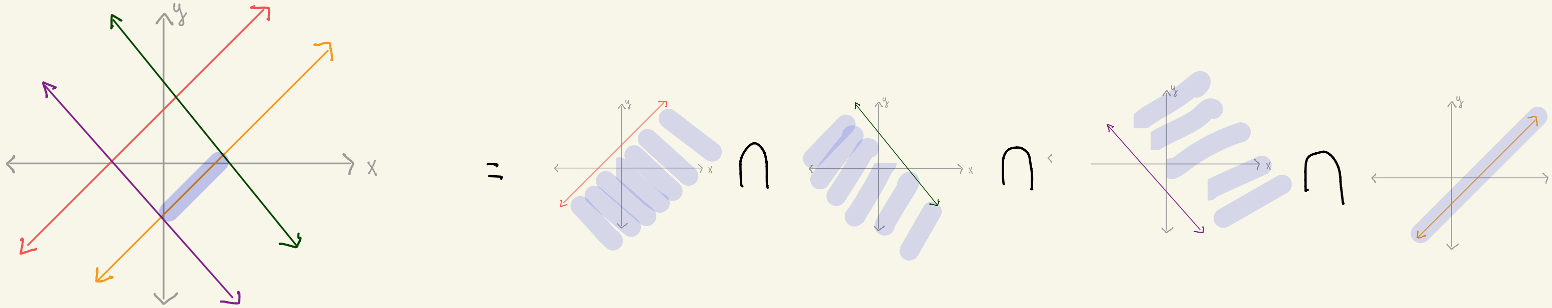
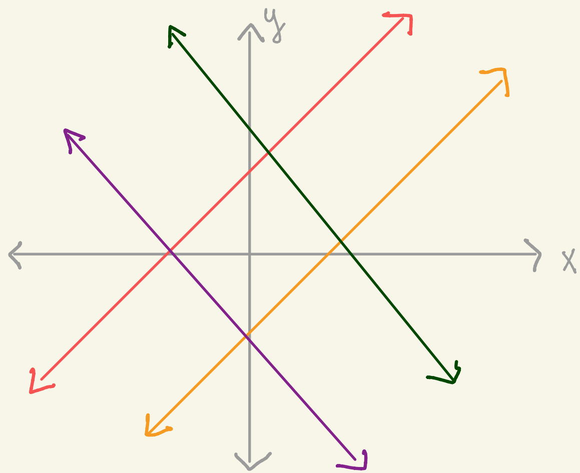




The faces of an arrangement are obtained by picking one such subset for each hyperplane, then taking the intersection.



Faces of Hyperplane Arrangements



The above arrangement has 25 faces

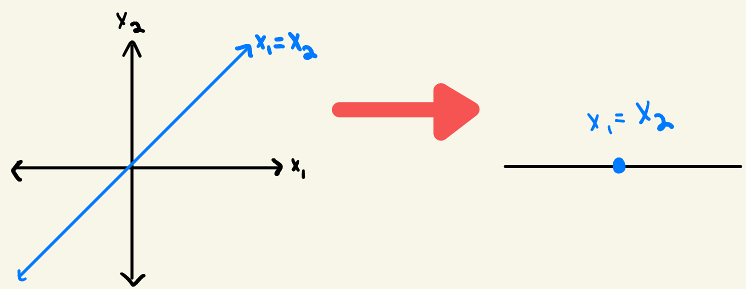
- 9 two-dimensional faces (chambers)
- 12 one-dimensional faces
- 4 zero-dimensional faces

The Braid Arrangement

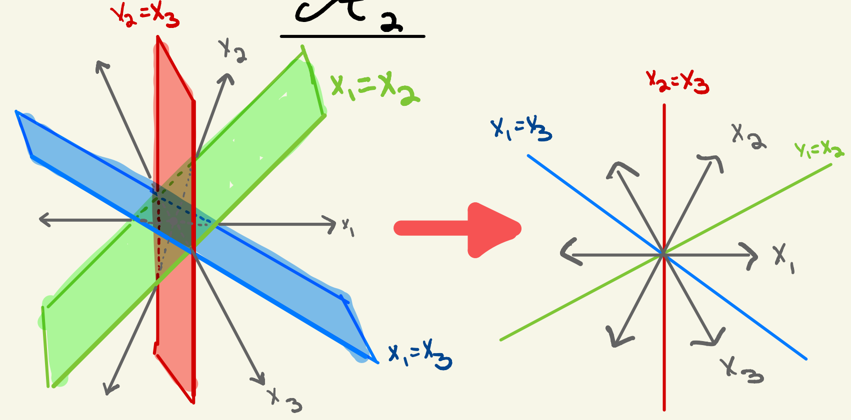
The Braid Arrangement $\mathcal{A}_{n-1}$ is the collection of hyperplanes

$$\{x_i = x_j \mid 1 \le i < j \le n\} \subseteq \mathbb{R}^n.$$

$\mathcal{A}_1$

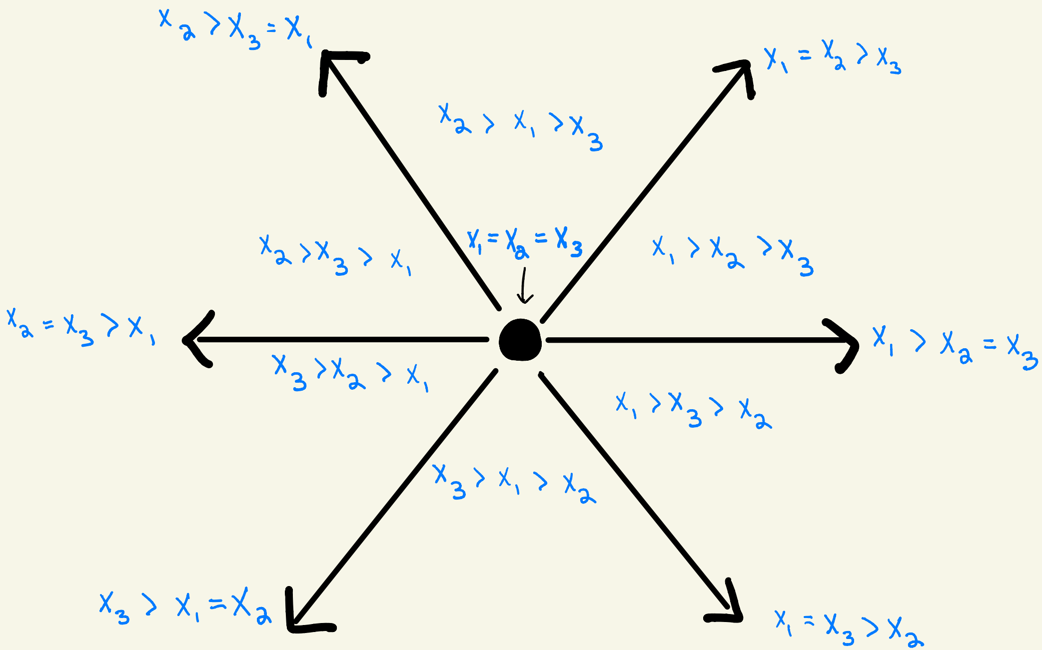


$\mathcal{A}_2$



Faces of the Braid Arrangement

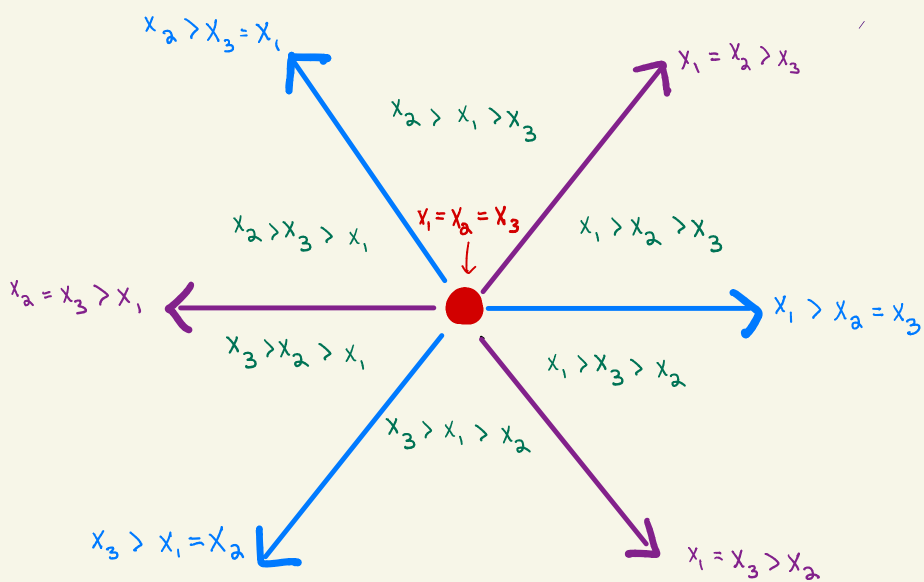
The faces of $\mathcal{A}_{n-1}$ are indexed by ordered set partitions of $\{1,2,\ldots,n\}$



Let $\mathcal{F}_n$ denote the set of faces of $\mathcal{A}_{n-1}$.

The Action of S_n on $\mathcal{F}_n$

The symmetric group S_n acts on $\mathcal{F}_n$ by permuting coordinates.



There are 2^{n-1} S_n-orbits, indexed by integer compositions of n: $\alpha \vDash n$.

$\mathbb{C}\mathcal{F}_n$ as an S_n-representation

The vector space $\mathbb{C}\mathcal{F}_n := \text{span}_{\mathbb{C}}\{F \in \mathcal{F}_{n-1}\}$ is an S_n-representation.

Recall

- For a finite group G, a representation of G consists of a:
  - $\mathbb{C}$-Vector space V
  - Linear action $G \circlearrowright V$
- We "understand" a G-representation by decomposing it into a direct sum of irreducible G-representations.

Irreducible representations of S_n are indexed by partitions $\lambda \vdash n$!

χ^λ

Writing S_n-representations as Symmetric Functions

Let $\bigoplus_{n\geq 0} \mathrm{Char}_n$ denote the ring of characters of S_n

product = induction product

$$\chi \otimes \psi \Big\uparrow_{S_m \times S_r}^{S_{m+r}}$$

S_m- character

S_r- character

There is an inner-product preserving ring isomorphism:

$$\mathrm{ch} : \bigoplus_{n\geq 0} \mathrm{Char}_n \longrightarrow \Lambda$$ ring of symmetric functions

$$\mathbb{1}_{S_n} \longmapsto h_n$$ "complete homogeneous symmetric function"

$$\chi^\lambda \longmapsto s_\lambda$$ "Schur function"

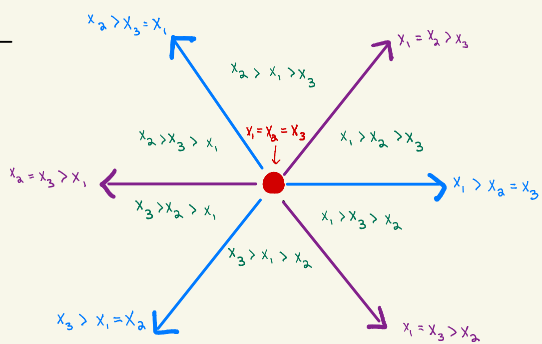
$\mathbb{C}\mathcal{F}_n$ as an S_n-representation

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, let $h_\alpha := h_{\alpha_1} \cdot h_{\alpha_2} \cdot \ldots \cdot h_{\alpha_k}$.

$$\mathrm{ch}(\mathbb{C}\mathcal{F}_n) = \sum_{\alpha \models n} h_\alpha = \sum_{\alpha \models n} K_{\lambda,\alpha}\, s_\lambda .$$

Kostka numbers

Example



$$\mathrm{ch}(\mathbb{C}\mathcal{F}_3) = h_3 + h_{21} + h_{12} + h_{111}$$

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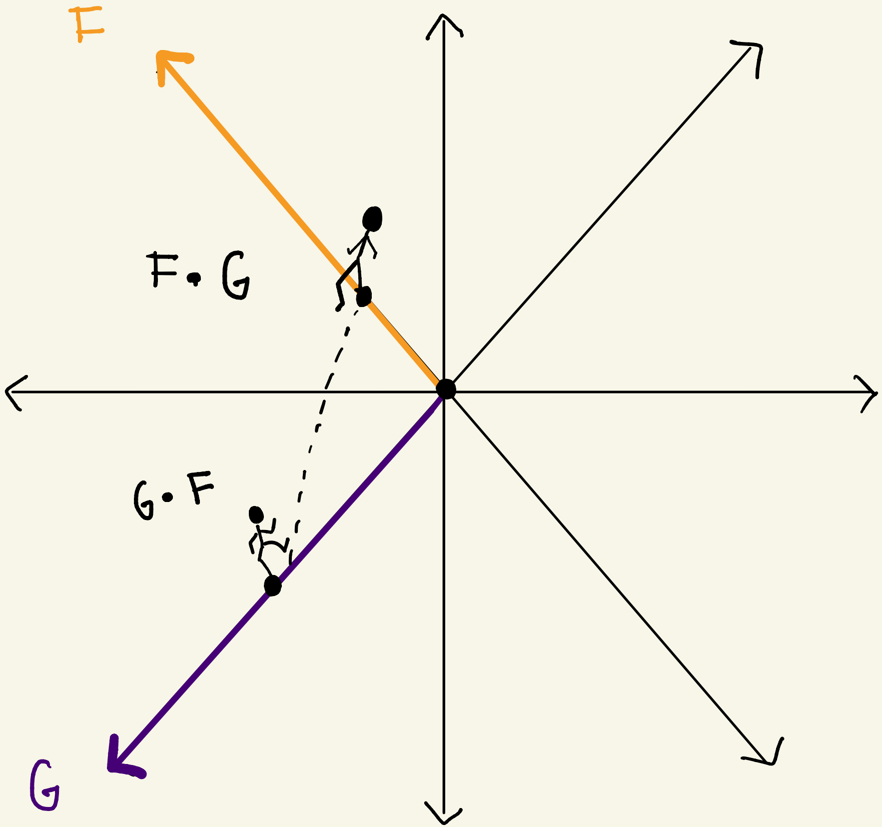
Face Multiplication



- The faces of a hyperplane arrangement form a noncommutative semigroup.

- If the arrangement is central, the faces form a monoid.

The Face Algebra

The vector space $\mathbb{C}\mathcal{F}_n$ is actually an algebra called the face algebra.

The face algebra is studied ...

1.  In the context of Markov Chains / Card Shuffling 

   Bidigare, Brown, Diaconis, Denham, Hanlon, Lafrenière, Reiner, Rockmore Saliola, Uyemura-Reyes, Welker, and more

2. As an algebra with combinatorial representation theory:

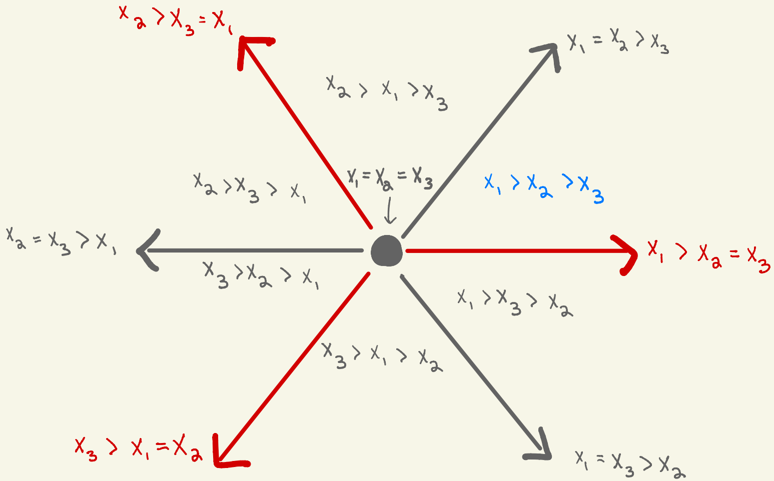
   Aguiar-Mahajan, Bastidas, Bidigare, Saliola, Schocker, and more

3. For its connections to Solomon's descent algebra

   Aguiar-Mahajan, Saliola, Schocker, and more

The face algebra + Shuffling

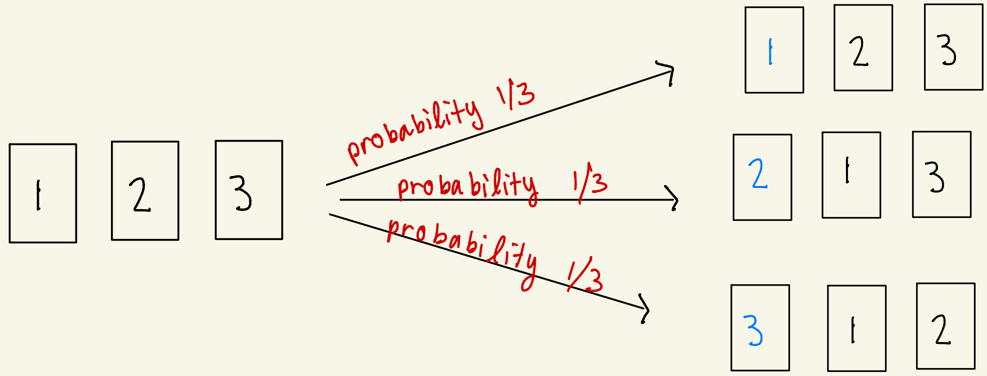
Let $y = 1/3(x_1 > x_2 = x_3) + 1/3(x_3 > x_1 = x_2) + 1/3(x_2 > x_3 = x_1)$.

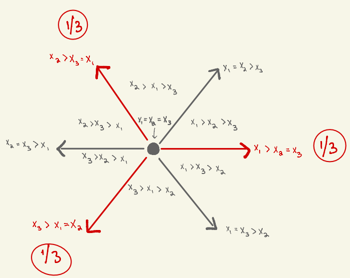


Then,

$$y \cdot (x_1 > x_2 > x_3) = \frac{1}{3}(x_1 > x_2 > x_3) + \frac{1}{3}(x_3 > x_1 > x_2) + \frac{1}{3}(x_2 > x_1 > x_3).$$

Random-to-top shuffling





$$y \cdot (x_1 > x_2 > x_3)$$

$$\parallel$$

$$\tfrac{1}{3}(x_1 > x_2 > x_3) + \tfrac{1}{3}(x_2 > x_1 > x_3) + \tfrac{1}{3}(x_3 > x_1 > x_2)$$

Random Walks + Shuffling in the Face Algebra

The subspace of chambers (which can be thought of as the subspace of decks of cards on $\{1,2,\ldots,n\}$) is closed under multiplication by $\mathbb{C}\mathcal{F}_n$.

Bidigare – Hanlon – Rockmore '99:

- Several popular Markov chains can be modeled within $\mathbb{C}\mathcal{F}_n$ by

$$\begin{pmatrix} \text{some elt} \\ \text{of } \mathbb{C}\mathcal{F}_n \end{pmatrix} \circlearrowright (\text{Chamber subspace}).$$

- Developed uniform formulas involving the combinatorics of the arrangement to compute the eigenvalues of such Markov chains

of probabilistic interest for bounding mixing time

Example: Eigenvalues of Random-to-top Shuffling

Random-to-top as a matrix on $\mathbb{C} S_n$:

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\
\begin{array}{c} 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} &
\begin{pmatrix}
1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\
0 & 1/3 & 0 & 0 & 1/3 & 1/3 \\
1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\
1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\
0 & 0 & 1/3 & 1/3 & 0 & 1/3
\end{pmatrix}
\end{array}
$$

$n = 3$

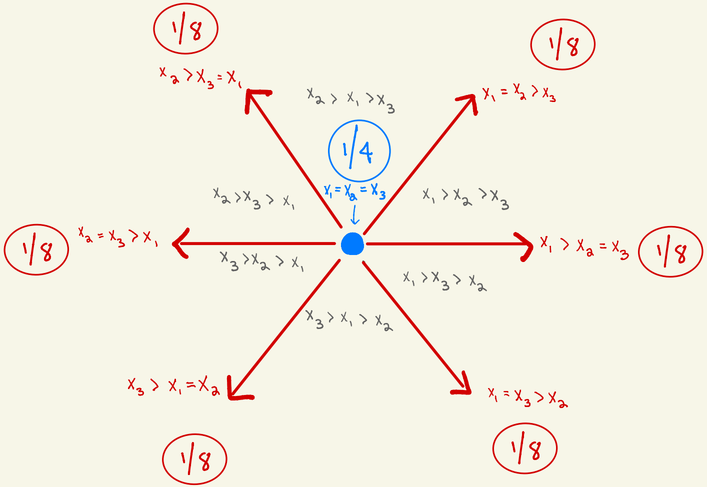
Theorem (Phatarfod '91, proof simplified by Bidigare-Hanlon-Rockmore approach)

- After scaling by n, the eigenvalues of random-to-top on $\mathbb{C} S_n$ are

$$0, 1, 2, \ldots, n-2, n.$$

- The multiplicity of the eigenvalue i is $\#\{\pi \in S_n : \pi \text{ has exactly } i \text{ fixed points}\}$.

Riffle Shuffling



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Solomon's descent algebra

- The descent set $\mathrm{Des}(\sigma)$ of a permutation $\sigma \in S_n$ consists of the positions $1 \le i \le n-1$ for which $\sigma(i) > \sigma(i+1)$.

  ↳ $\mathrm{Des}(\underline{7}\,\underline{5}\,1\,\underline{3}\,2\,\underline{6}\,4\,8\,9) = \{1,2,4,6\}$

  one-line notation

- For $J \subseteq \{1,2,\ldots,n-1\}$, define

$$x_J := \sum_{\substack{\sigma \in S_n : \\ \mathrm{Des}(\sigma) = J}} \sigma \quad \in \quad \mathbb{C} S_n .$$

- The 2^{n-1}-dim'l vector space $\mathrm{span}_{\mathbb{C}} \{x_J \mid J = \{1,2,\ldots,n-1\}\}$ is closed under multiplication, so forms a subalgebra of $\mathbb{C}S_n$ called

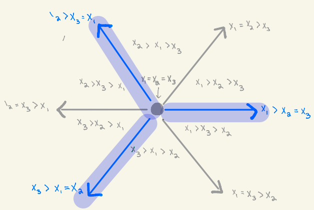
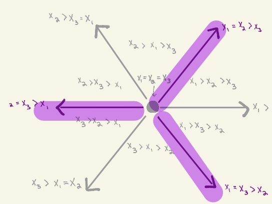
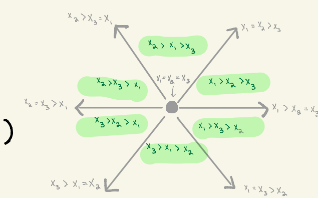
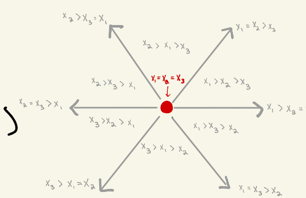
  Solomon's descent algebra: Σ_n.

The S_n-invariant subalgebra of $\mathbb{C}\widetilde{\mathcal{F}}_n$

- The action of S_n on $\mathbb{C}\widetilde{\mathcal{F}}_n$ is by algebra automorphisms

- The S_n-invariant subalgebra $(\mathbb{C}\widetilde{\mathcal{F}}_n)^{S_n}$ consists of the elements fixed under the action of S_n.

<u>Example</u> $(\mathbb{C}\widetilde{\mathcal{F}}_4)^{S_4} =$

$\operatorname{span}_{\mathbb{C}} \Bigg\{$  ,  ,  ,  $\Bigg\}$

The key connection

Thm (Bidigare): The descent algebra is anti-isomorphic to the S_n-invariant subalgebra of the face algebra:

$$\Sigma_n^{\mathrm{opp}} \cong \left(\mathbb{C}\mathcal{F}_n\right)^{S_n}.$$

↳ A useful way to view Σ_n: Saliola computed its quiver using this embedding

Representation Theory of Finite Dimensional Algebras

A representation of a finite dimensional algebra consists of a

- $\mathbb{C}$- vector space V, and a
- linear action* $A \circlearrowright V$

Example:

The face algebra is a (right) representation of the descent algebra.

$\mathbb{C}\mathcal{F}_n \circlearrowleft \Sigma_n$ via $y \circlearrowleft x := \Phi^{-1}(x) \cdot y$ Bidigare anti-isomorphism

---

Unlike group representations over $\mathbb{C}$, algebra representations do not always decompose into a direct sum of irreducible representations.

$\hookrightarrow$ I.e. A is not semisimple

n

How do we "understand" representations of nonsemisimple algebras?

One Option: filtrations of an A-representation V by subrepresentations,

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_k = V \quad \text{with each } V_i/V_{i-1} \text{ irreducible.}$$

Up to isomorphism, the multiset of irreducibles $\{V_i/V_{i-1}\}$ is independent of the filtration!

Composition Multiplicity of M in V $= [V : M] =$ # of times M (up to isomorphism) appears in $\{V_i/V_{i-1}\}$.

(Right) Representation Theory of Σ_n

- Studied in depth by Garsia–Reutenauer, generalized to other types by F. Bergeron, N. Bergeron, Howlett, Taylor

- Σ_n is not semisimple

- Integer partitions of n $\longleftrightarrow$ Irreducible representations $\longleftrightarrow$ Idempotents in a complete family of primitive, orthogonal idempotents

  λ $\qquad$ M_λ $\qquad$ E_λ

  one-dimensional!

- To understand a Σ_n-rep'n V:

  ↳ Count the composition multiplicity $[V : M_\lambda]$ for each λ

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The Face Algebra as a Simultaneous Representation

We've seen: $S_n \curvearrowright \mathbb{C}\mathcal{F}_n$.

But also: $\mathbb{C}\mathcal{F}_n \curvearrowleft \Sigma_n$

How about: $S_n \curvearrowright \mathbb{C}\mathcal{F}_n \curvearrowleft \Sigma_n$?

Isotypic components: Generalizations of invariant subalgebras

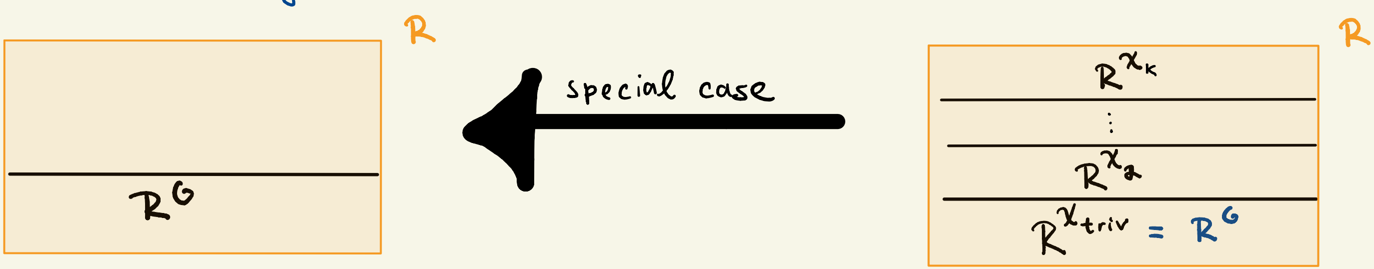
Finite group G acting on a fin. dim. $\mathbb{C}$-algebra R by algebra homomorphisms.

For an irreducible G-representation χ:

R^{χ} = χ-isotypic component

= direct sum of all copies of χ in any decomp of R

Invariant Subalgebra | G- isotypic components



Each isotypic subspace R^{χ} is an R^G-module.

My Question

How does each S_n-isotypic component $(\mathbb{C}\mathcal{F}_n)^{\chi^\nu}$ look as a representation of Σ_n?

$(\mathbb{C}\mathcal{F}_n)^{\chi_\mu}$	? ?	$\circlearrowleft \Sigma_n$
$\vdots$		
$(\mathbb{C}\mathcal{F}_n)^{\chi_\lambda}$	? ?	$\circlearrowleft \Sigma_n$
$(\mathbb{C}\mathcal{F}_n)^{\chi_\nu}$	? ?	$\circlearrowleft \Sigma_n$
$(\mathbb{C}\mathcal{F}_n)^{S_n} \cong \Sigma_n^{opp}$		$\circlearrowleft \Sigma_n$

Motivation

- Standard question in classical invariant theory
- Natural Extension of Bidigare's work, who studied $(\mathbb{C}\mathcal{F}_n)^{\chi^\nu}$ for χ^ν
  - the trivial representation χ^n
  - the sign representation χ^{1^n}
- New examples of Σ_n-representations

A first approximation

Proposition As Σ_n-representations

$$(\mathbb{C}\mathcal{F}_n)^{\chi_\nu} = \bigoplus_{\mu \text{ dominates } \nu} (\mathbb{C}\mathcal{F}_n E_\mu)^{\chi_\nu}, \quad \text{with}$$

$$\dim_{\mathbb{C}} (\mathbb{C}\mathcal{F}_n E_\mu)^{\chi_\nu} = \#\left\{\underset{\alpha}{\text{compositions}} \,\middle|\, \alpha \text{ rearranges to } \mu\right\} \cdot \underbrace{\#SYT(\nu)}_{\text{Standard Young tableaux}} \cdot \underbrace{K_{\nu,\mu}}_{\text{Kostka numbers}}$$

Abuse of notation*: viewing E_μ in $(\mathbb{C}\mathcal{F}_n)^{S_n}$ rather than Σ_n

A first approximation

$\mathbb{C}\mathcal{J}_4$

$\#\alpha \vDash n \sim \mu$

$\#SYT(\nu)$

$K_{\nu,\mu}$

S_n-isotypic subspaces (ν)	$\mathbb{C}\mathcal{J}_n E_4$	$\mathbb{C}\mathcal{J}_n E_{31}$	$\mathbb{C}\mathcal{J}_n E_{22}$	$\mathbb{C}\mathcal{J}_n E_{211}$	$\mathbb{C}\mathcal{J}_n E_{1111}$
χ^{1111}					$1 \cdot 1 \cdot 1$
χ^{211}				$3 \cdot 3 \cdot 1$	$1 \cdot 3 \cdot 3$
χ^{22}			$1 \cdot 2 \cdot 1$	$3 \cdot 2 \cdot 1$	$1 \cdot 2 \cdot 2$
χ^{31}		$2 \cdot 3 \cdot 1$	$1 \cdot 3 \cdot 1$	$3 \cdot 3 \cdot 2$	$1 \cdot 3 \cdot 3$
$\chi^{4} = \chi^{triv}$	$1 \cdot 1 \cdot 1$	$2 \cdot 1 \cdot 1$	$1 \cdot 1 \cdot 1$	$3 \cdot 1 \cdot 1$	$1 \cdot 1 \cdot 1$

μ

A Better Answer

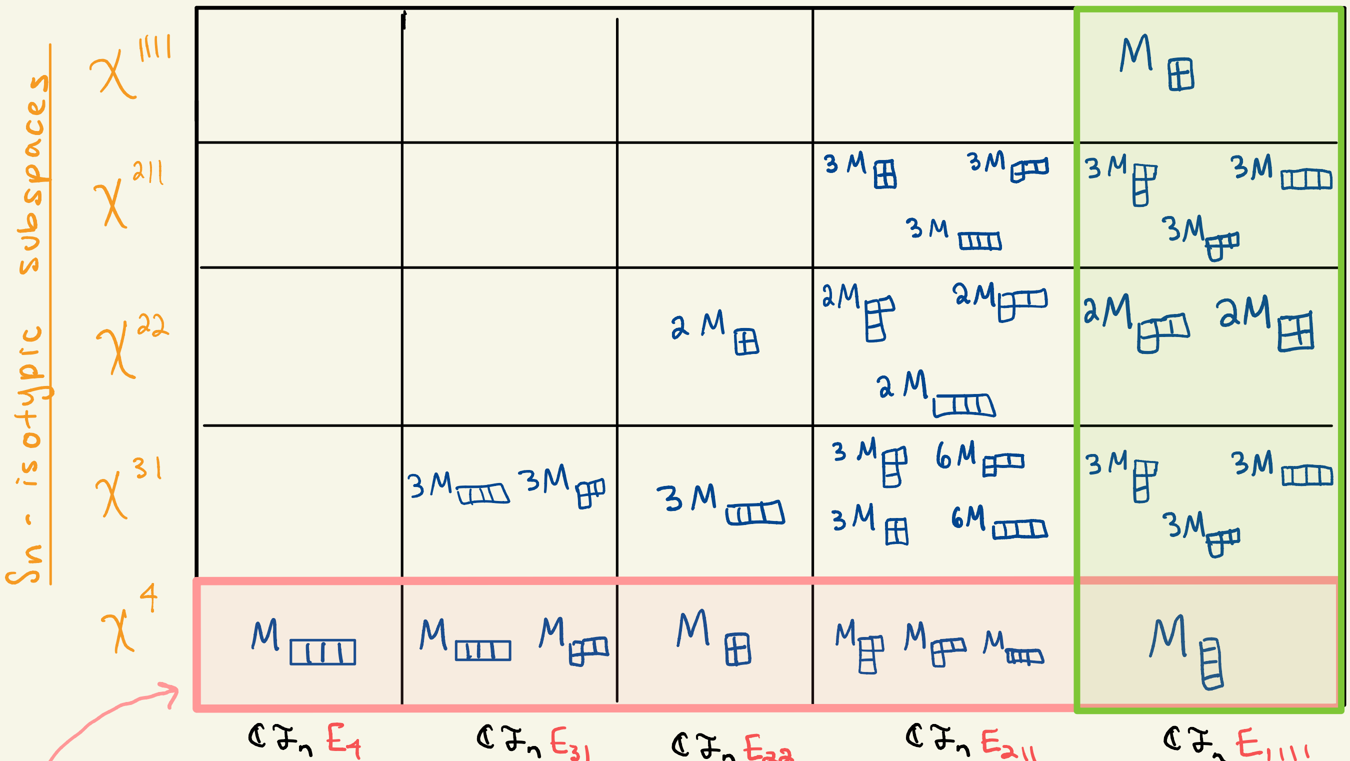
Fill in submodules with Σ_n-composition factors M_λ

S_n-isotypic subspaces (ν)

	$\mathbb{C}\mathcal{J}_n E_4$	$\mathbb{C}\mathcal{J}_n E_{31}$	$\mathbb{C}\mathcal{J}_n E_{22}$	$\mathbb{C}\mathcal{J}_n E_{211}$	$\mathbb{C}\mathcal{J}_n E_{1111}$
χ^{1111}					$M_{(2,2)}$
χ^{211}				$3M_{(2,2)}$ $3M_{(3,1)}$ $3M_{(4)}$	$3M_{(2,1,1)}$ $3M_{(4)}$ $3M_{(3,1)}$
χ^{22}			$2M_{(2,2)}$	$2M_{(2,1,1)}$ $2M_{(3,1)}$ $2M_{(4)}$	$2M_{(3,1)}$ $2M_{(2,2)}$
χ^{31}		$3M_{(4)}$ $3M_{(3,1)}$	$3M_{(4)}$	$3M_{(2,1,1)}$ $6M_{(3,1)}$ $3M_{(2,2)}$ $6M_{(4)}$	$3M_{(2,1,1)}$ $3M_{(4)}$ $3M_{(3,1)}$
χ^{4} ($\chi^{\text{"triv"}}$)	$M_{(4)}$	$M_{(4)}$ $M_{(3,1)}$	$M_{(2,2)}$	$M_{(2,1,1)}$ $M_{(3,1)}$ $M_{(4)}$	$M_{(1,1,1,1)}$

μ

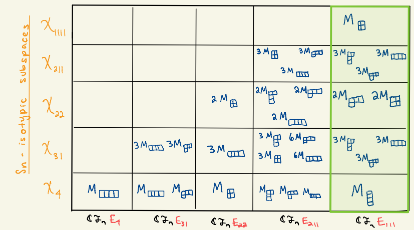
Previously Known



Rightmost column: Uyemura-Reyes's shuffling rep'ns

Bottom row: Garsia – Reutenauer's Cartan invariants of Σ_n

Uyemura-Reyes's Shuffling Representations



In terms of "higher Lie" symmetric functions $\mathcal{L}_\lambda$.

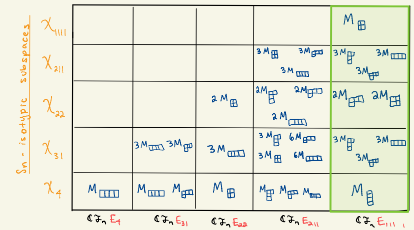
When $GL_n(\mathbb{C})$ acts on $\mathbb{C}^n =: V$,

$\mathcal{L}_\lambda(x_1, x_2, \ldots, x_n)$ is the character / trace of $\begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{bmatrix}$ on $\mathrm{Lie}_\lambda(V)$, where:

$$\underset{\text{tensor algebra}}{T(V)} \cong_{GL(V)\text{-reps}} \mathrm{Sym}\big(\underbrace{\mathrm{Lie}(V)}_{\text{free Lie algebra}}\big)$$

$$= \bigoplus_{\lambda = 1^{m_1} 2^{m_2} 3^{m_3}} \underbrace{\mathrm{Sym}^{m_1}(V) \otimes \mathrm{Sym}^{m_2}([V,V]) \otimes \mathrm{Sym}^{m_3}([[V,V],V]) \otimes \ldots}_{\overset{\shortparallel}{=:}\ \mathrm{Lie}_\lambda(V)}.$$

Uyemura-Reyes's Shuffling Representations

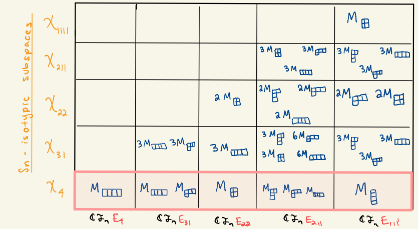


Thrall's Problem

Positively expand $\mathcal{L}_\lambda = \sum c_{\mu,\lambda} s_\mu$.

Thm (Uyemura-Reyes, 2002)

$$\left[(\mathbb{C}\tilde{\mathcal{J}}_n)^{\chi_\nu}_{E_{1^n}} : M_\lambda \right] = \# SYT(\nu) \langle s_\nu , \mathcal{L}_\lambda \rangle$$



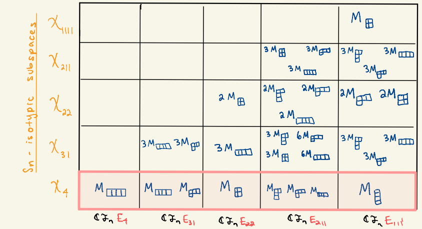
Garsia-Reutenauer's Cartan Invariants of Σ_n

- The Σ_n-representations $(\mathbb{C}\mathcal{F}_n)^{S_n} E_\mu$ are important representations called the

  projective indecomposables of Σ_n.

  ↳ The composition multiplicities $[(\mathbb{C}\mathcal{F}_n)^{S_n} E_\mu : M_\lambda]$ are called the Cartan invariants of Σ_n.

Garsia-Reutenauer's Cartan Invariants of Σ_n



- The Σ_n - representations $(\mathbb{C}\mathcal{F}_n)^{S_n} E_\mu$ are important representations called the

  projective indecomposables of Σ_n.

  ↳ The composition multiplicities $[(\mathbb{C}\mathcal{F}_n)^{S_n} E_\mu : M_\lambda]$ are called the Cartan invariants of Σ_n.

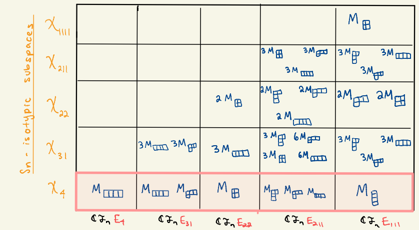
The Cartan invariants of Σ_n have a beautiful formula!

- A word on $\{1, 2, \ldots\}$ is a Lyndon word if it is strictly lexicographically smaller than all of its cyclic reorderings.

Example 29299

Non-example 2929

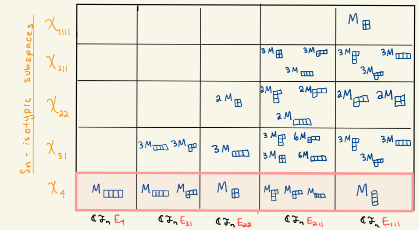
Garsia - Reutenauer's Cartan Invariants of Σ_n



- Every word has a unique factorization into lexicographically weakly decreasing Lyndon words

Word w	Lyndon Factorization	Lyndon Type (w)
5 6 1 4 3 1 1 2 3 6	$\underbrace{(56)}_{11}\ \underbrace{(143)}_{8}\ \underbrace{(11236)}_{13}$	$(13, 11, 8)$
1 2 1 1 2 3	$(12)\ (1123)$	$(7, 3)$

Garsia-Reutenauer's Cartan Invariants of Σ_n



- Every word has a unique factorization into lexicographically weakly decreasing Lyndon words

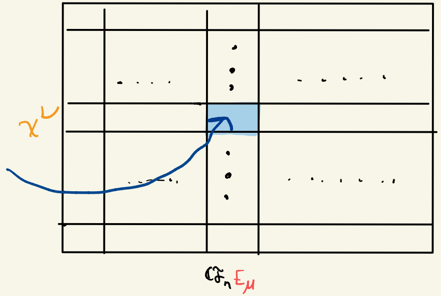
Word w	Lyndon Factorization	Lyndon Type (w)
5614311236	$\underbrace{(56)}_{11}\,\underbrace{(143)}_{8}\,\underbrace{(11236)}_{13}$	$(13, 11, 8)$
121123	$(12)\,(1123)$	$(7, 3)$

Thm (Garsia-Reutenauer, 1989)

$$\left[(\mathbb{C}\mathcal{F}_n)^{S_n} E_\mu : M_\lambda \right] = \#\left\{ \begin{array}{l} \text{compositions } \alpha \\ \text{rearranging to } \mu \end{array} \;\middle|\; \text{LyndonType}(\alpha) = \lambda \right\}.$$

The Full Table

How many M_λ's to draw?



Thm: (C., 2024)

$$\left[(\mathbb{C}\mathfrak{F}_n)^{\chi_\nu} E_\mu : M_\lambda\right] = \#\mathrm{SYT}(\nu)\cdot\left\langle s_\nu \,,\, \text{coefficient of } \underline{x}_\lambda \underline{y}_\mu \text{ in } F\right\rangle,$$

where
$$F = \prod_{\substack{\text{Lyndon} \\ \text{words} \\ w}} \sum_{\substack{\text{Partitions} \\ \tau}} \underline{x}_{\tau\cdot|w|} \left(\underline{y}_w\right)^{|\tau|} \mathcal{L}_\tau \left[h_w\right].$$

Notation

- $\underline{x}_\lambda := x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_k}$
- $|\nu|$, $|w|$:= sum of the parts of ν and the letters of w.
- $\nu \cdot k$:= scaling of the parts of ν by k

Proof Ingredients

Understanding a Σ_n-representation

$$\left[(\mathbb{C}\mathcal{F}_n)^{\chi_\nu} E_\mu : M_\lambda \right] ?$$

Understanding an S_n-representation

$$E_\lambda \mathbb{C}\mathcal{F}_n E_\mu ?$$

Poset Topology

$$\bigoplus_{\substack{[X \leq Y] \\ X \in \mu \\ Y \in \lambda}} \widetilde{H}_{top}\left(\Pi_n(X,Y)\right) \otimes \mathrm{Det}(Y) \otimes \mathrm{Det}(X) \Big\uparrow_{\mathrm{Stab}_{S_n}(X) \cap \mathrm{Stab}_{S_n}(Y)}^{S_n} ?$$

homology of intervals in the lattice of set partitions

Symmetric Functions

$$\prod_{\substack{\text{Lyndon} \\ \text{words} \\ w}} \sum_{\substack{\text{Partitions} \\ \tau}} \underline{x}_{(\tau \cdot |w|)} \left(\underline{y}_w\right)^{|\tau|} \mathcal{L}_\tau[h_w].$$

Thank you!

For details : arXiv 2404.00536