Multiplying quantum Schubert polynomials using combinatorics

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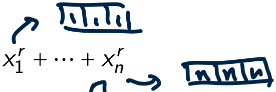
The problem

Problem proposed by Frank Sottile during the Combinatorics seminar at the Fields Institute

I'd like to know how to multiply ***efficiently*** *quantum Schubert polynomials and I would like to try a particular idea I have.*

WHAT DO I MEAN BY *multiplying*?

- Schubert polynomials are generalizations of Schur polynomials
- Symmetric polynomials world
  - Power sum (multiplicative) basis: $p_r = x_1^r + \cdots + x_n^r$
  - Schur basis: $s_\lambda = \sum_T x^T$



$$p_{(r)} = s_{(r)} - s_{(r-1,1)} + \cdots + (-1)^{r-1} s_{(1^r)}$$

Murnaghan-Nakayama rule [Mur'37, Nak'40]

$$p_r(x) \cdot s_\lambda(x) = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu(x),$$

where the sum ranges over all μ such that μ/λ is a border-strip with r boxes.

How do we get there?

$p_{(r)} = s_{(r)} - s_{(r-1,1)} + \cdots + (-1)^{r-1} s_{(1^r)}$ $\rightsquigarrow$ $p_r(x) \cdot s_\lambda(x) = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu(x)$

- Multiplication by a box $s_{\square}$
- Multiplication by a column $s_{(1,1,1,1)}$ and a row $s_{(4)}$
- Multiplication by a hook $s_{(3,1,1,1)}$
- Hope that your combinatorial model behaves well with the alternating signs so that *only one object survives*.

Let's do it for Schubert polynomials

Flag variety $\mathbb{F}\ell_n$ & its cohomology $H^*\mathbb{F}\ell_n$

- GL_n acts on flags: for $g \in GL_n$,

$$F_1 \subset F_2 \subset \cdots \subset F_n \Longrightarrow g(F_1) \subset g(F_2) \subset \cdots \subset g(F_n)$$

GL_n: group of invertible linear transformations of $\mathbb{C}^n$

- **Flag variety** $\mathbb{F}\ell_n$: GL_n-orbit of a flag F $\qquad \mathbb{F}\ell_n = GL_n/Stab$

- **Schubert cells** X_w: orbits in $\mathbb{F}\ell_n$ under the action of $Stab$
  $X_w = \{bw(F) \mid b \in Stab\} \qquad (w \in S_n)$

- Cohomology of the flag variety $H^*\mathbb{F}\ell_n$

- **Schubert classes** $[\overline{X_w}]$: distinguished basis of $H^*\mathbb{F}\ell_n$.

Product: (generic) intersection $[\overline{X_v}] \cdot [\overline{X_w}] = [\overline{X_v} \cap \overline{X_w}]$

Schubert polynomials – Algebraic geometry

Borel proved that

$$H^*\mathbb{F}\ell_n \cong \mathbb{Z}[x_1,\ldots,x_n]\big/\langle e_1,\ldots,e_n\rangle,$$

e_i : elementary symmetric polynomial in the alphabet $\{x_1,\ldots,x_n\}$

Algebraic geometry		Algebraic combinatorics
intersection	$\longleftrightarrow$	multiplication
$\left[\overline{X_w}\right]$ Schubert classes	$\longleftrightarrow$	$\mathfrak{S}_w$ Schubert polynomials

Schubert polynomials – recursive definition

For $w_0 = (n, n-1, \ldots, 1)$, $\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^{1}$

For $w \neq w_0$ with $w(i) < w(i+1)$, $\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i}$, where

$$\partial_i = \frac{P - s_i P}{x_i - x_{i+1}},$$

divided differences operator

$s_i P(x_1 \ldots x_n) = P(x_1, \ldots, x_{i+1}, x_i \ldots x_n)$

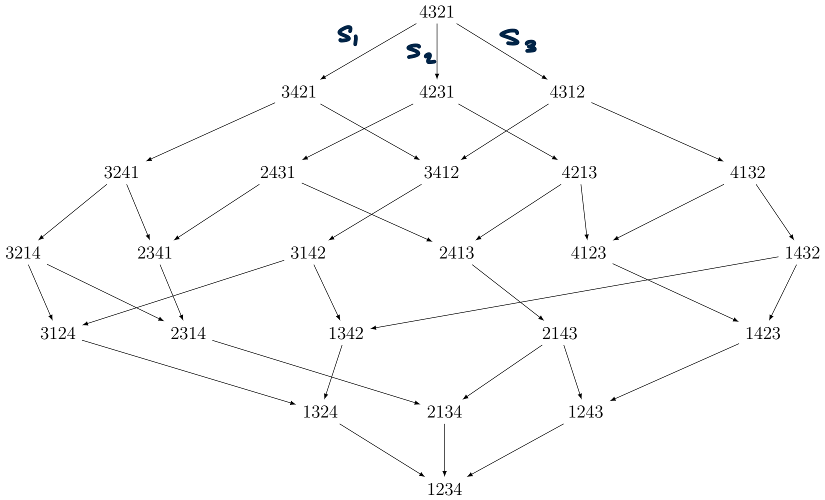
<u>Example:</u>

$\mathfrak{S}_{54321} = x_1^4 x_2^3 x_3^2 x_4$ $\qquad$ $\mathfrak{S}_{24531} = ?$

$$54321 \xrightarrow{\partial_1} 45321 \xrightarrow{\partial_3} 45231 \xrightarrow{\partial_2} 42531 \xrightarrow{\partial_1} 24531$$

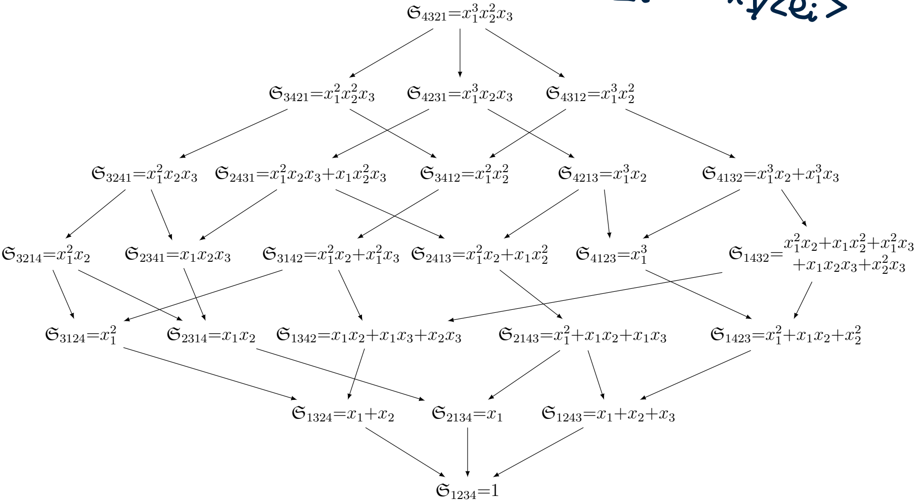
$$\mathfrak{S}_{24531} = \partial_1 \partial_2 \partial_3 \partial_1 \mathfrak{S}_{54321} = x_1 x_2^2 x_3^2 x_4 + x_1^2 x_2^2 x_3 x_4 + x_1^2 x_2 x_3^2 x_4$$

Schubert polynomials [Lascoux - Schützenberger (1982)]

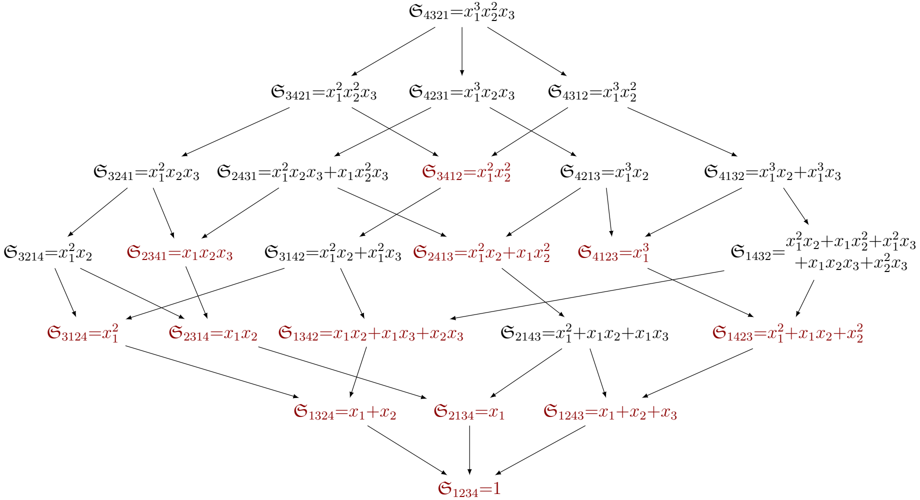


Schubert polynomials [Lascoux - Schützenberger (1982)]

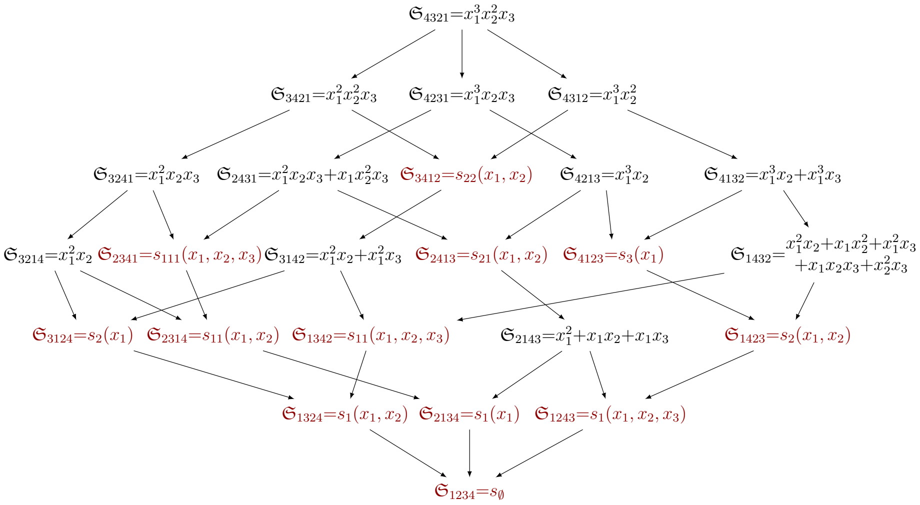
$\mathbb{Z}[x_1, \ldots x_n]/\langle e_i \rangle$



Schubert polynomials [Lascoux - Schützenberger (1982)]



Schubert polynomials [Lascoux - Schützenberger (1982)]



Schur polynomials as Schubert polynomials

Look at the permutations with one single descent in position k.

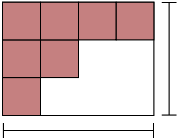
$$w = (2,4,7 \mid 1,3,5,6) \longrightarrow \text{ Descent in position } k = 3$$

k=3

$\lambda = (4,2,1)$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{pmatrix}$$

1 2 4



$k = 3$

$n - k = 4$

Every partition $\lambda \subset R_{k,n-k}$ corresponds to a permutation in S_n with a unique descent in position k, and we denote it by $v(\lambda,k)$.

$$\boxed{\mathfrak{S}_{v(\lambda,k)}(x) = s_\lambda(x_1,\ldots,x_k)}$$

Open problem

Find combinatorial rule to compute the coefficients d_{uv}^{w} in

$$\mathfrak{S}_u(x) \cdot \mathfrak{S}_v(x) = \sum_{w} d_{uv}^{w} \mathfrak{S}_w(x)$$

- Avoid linear algebra and computing the polynomials.
- Compute the coefficients d_{uv}^{w} using information from u, v, w.
- The answer is know for several particular cases.

Open problem

Combinatorial rule for d_{uv}^w

$$\mathfrak{S}_u(x)\cdot\mathfrak{S}_v(x)=\sum_w d_{uv}^w\mathfrak{S}_w(x)$$

Monk's Rule

For $u\in S_n$ and $k<n$,

$$\underbrace{\mathfrak{S}_{(k,k+1)}(x)}_{S_n}\cdot\mathfrak{S}_u(x)=\sum_v\mathfrak{S}_v(x)$$

k-Bruhat order in S_n

$u \lessdot_k u(i,j) = v$ iff

$$\begin{cases} i\le k<j \quad \& \quad u(i)<u(j) \\ \nexists l: i<l<j \;\&\; u(i)<u(l)<u(j) \\ \Leftrightarrow \ell(u(i,j))=\ell(u)+1\end{cases}$$

Example: $n=7,\ k=4$

$2,5\notin[6,7]$

$3146|257 \lessdot_k 3147|256$

$3146|257 \lessdot_k 3156|247$

$6,2\notin[4,5]$

$$\mathfrak{S}_{(4,5)}\cdot\mathfrak{S}_{3146257}=\mathfrak{S}_{3147256}+\mathfrak{S}_{3156247}+\cdots$$

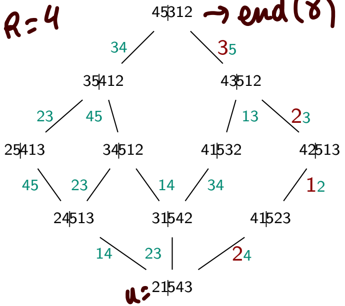
Multiplying by a hook

Theorem [Sottile, '96]

Let $u \in S_n$, $l \leq k$, and $m \leq n-k$. Then

$$\mathfrak{S}_u(x) \cdot s_{(m,1^{l-1})}(x_1,\ldots,x_k) = \sum_{\gamma} \mathfrak{S}_{\text{end}(\gamma)}(x)$$

summing over all *peakless chains* of height l and length $m+l-1$.



Peakless chain of height l & length r

$u \lessdot_k u(i,j) \rightsquigarrow u \xrightarrow{u(i)} u(i,j)$

$\gamma = \left(u \xrightarrow{a_1} u_1 \xrightarrow{a_2} \cdots \xrightarrow{a_r} u_r =: \text{end}(\gamma) \right)$

$a_1 > \cdots > a_l < a_{l+1} < \cdots < a_r$

$2 > 1 < 2 < 3$

$\mathfrak{S}_{21543}(x)s_{31}(x_1,x_2) = \mathbf{1}\mathfrak{S}_{45312}(x) + \cdots$

Another example

Theorem [Sottile (1996)]

$$\mathfrak{S}_u(x) \cdot s_{(m,1^{l-1})}(x_1,\ldots,x_k) = \sum_{\gamma} \mathfrak{S}_{\mathrm{end}(\gamma)}(x)$$

$$\mathfrak{S}_{135246}(x)s_3(x_1,x_2,x_3) = \mathbf{1}\mathfrak{S}_{246135}(x) + \cdots$$

$$\mathfrak{S}_{135246}(x)s_{21}(x_1,x_2,x_3) = \mathbf{2}\mathfrak{S}_{246135}(x) + \cdots$$

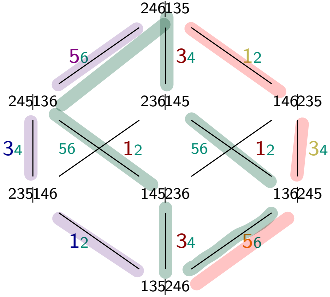
$$\mathfrak{S}_{135246}(x)s_{111}(x_1,x_2,x_3) = \mathbf{1}\mathfrak{S}_{246135}(x) + \cdots$$

$1 < 3 < 5$

$3 \vee 1 < 5$

$5 \vee 1 < 3$

$5 \vee 3 \vee 1$



Multiplying by a hook

Theorem [BBCSS (2022)]

Let $u \in S_n$, $l \leq k$ and $m \leq n-k$. Then

$$\mathfrak{S}_u(x) \cdot s_{(m,1^{l-1})}(x_1,\dots,x_k) \;=\; \sum \binom{s(\zeta)-1}{\mathrm{ht}(\zeta)-l} \mathfrak{S}_{\zeta u}(x)\,,$$

over minimal permutations $\zeta \in S_n$ s.t. $u \leq_k \zeta u$ and $\mathcal{L}(\zeta) = m+l-1$.

k-Bruhat order on S_n: $u \leq_k v$

$[u,v]_k \cong [e, \zeta := vu^{-1}]_k$

(e = id)

Grassmannian Bruhat order on S_n

$\eta \leq \zeta$ iff $\exists u \in S_n,\ k \in \mathbb{Z}_{>0}$ $s.t.$

$[\eta, \zeta]$ $\quad u \leq_k \eta u \leq_k \zeta u$

- Ranked order with $\mathcal{L}(\zeta) = \ell(\zeta u) - \ell(u)$
- $s(\zeta)$:= # cycles in the disjoint factorization of ζ
- $\mathrm{supp}(\zeta) = \{i \in [n] \mid \zeta(i) \neq i\}$ & $\mathrm{ht}(\zeta) := \#\{i \in [n] \mid i < \zeta(i)\}$
- ζ is minimal if $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - s(\zeta)$

M–N rule for $H^*\mathbb{F}\ell_n$ [BBCSS (2022)]]

Let $u \in S_n$. Then

$$\mathfrak{S}_u(x) \cdot p_r(x_1, \ldots, x_k) = \sum (-1)^{\mathrm{ht}(\zeta)+1} \mathfrak{S}_{\zeta u}(x),$$

summing over all minimal cycles $\zeta \in S_n$ s. t. $u \leq_k \zeta u$ and $\mathcal{L}(\zeta) = r$.

- Our proof uses the multiplication by a hook rule together with understanding when the coefficients are zero (unless unless $\mathcal{L}(\zeta) = r$ and ζ is minimal). For those initially non-zero coefficients, these are

$$(-1)^{\mathrm{ht}(\zeta)+1} \sum_{l=1}^{r} (-1)^{l-\mathrm{ht}(\zeta)} \binom{s(\zeta)-1}{\mathrm{ht}(\zeta)-l} = \begin{cases} 0, & \text{if } s(\zeta) \neq 1, \\ (-1)^{\mathrm{ht}(\zeta)+1}, & \text{if } s(\zeta) = 1. \end{cases}$$

- Our proof differs from those given by Morrison (2014) and by Morrison and Sottile (2018).

$q = \{q_1, \dots q_{n-1}\}$

q-WORLD

Algebraic geometry		Algebraic combinatorics
$\mathbf{q}H^*(F\ell_n)$ $H^*\mathbb{F}\ell_n \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, \dots, q_{n-1}]$	$\cong$	$\mathbb{Z}[x_1, \dots, x_n][q_1, \dots, q_{n-1}]/\langle E_1, \dots, E_n \rangle$

q-version of symm. polynomials.

- The q-**Schubert polynomials** $\mathfrak{S}_w^q(x) \in \mathbb{Z}[q, x]$ are the image of the q-Schubert classes under the isomorphism above.

Open problem

Find a combinatorial rule for $d_{uv}^w(q)$ in

$$\mathfrak{S}_u^q \star \mathfrak{S}_v^q = \sum_w d_{uv}^w(q) \mathfrak{S}_w^q$$

- q-Monk's rule
- **Multiplying by a q-hook**
- q-**Murnaghan–Nakayama's rule** p_R^q

Quantum k-Bruhat order in $S_n[q]$

- Classic case: $u \lessdot_k^q u(i,j)$ if

$$\begin{cases} i \leq k < j \quad \& \quad u(i) < u(j) \quad \& \\ \nexists l : i < l < j \quad \& \quad u(i) < u(l) < u(j) \\ \Leftrightarrow \ell(u(i,j)) = \ell(u) + 1 \end{cases}$$

what is in between is not in between

6|257

2,5 ∉ [6,7]

- Quantum case: $u \lessdot_k^q q_{ij} u(i,j)$ if

$q_i q_{i+1} \cdots q_{j-1}$

$\ell(q_i) = 2$

$$\begin{cases} i \leq k < j \quad \& \quad u(i) > u(j) \quad \& \\ \forall l : i < l < j \quad \& \quad u(i) > u(l) > u(j) \\ \Leftrightarrow \ell(u(i,j)) = \ell(u) + 1 + 2(i-j) \end{cases}$$

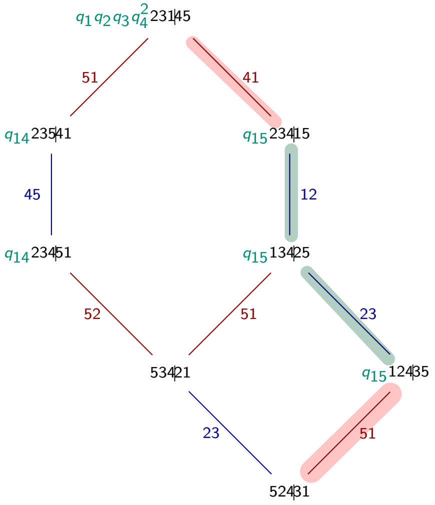
what is in between is in between

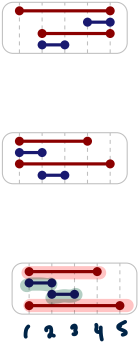
Example: For $n = 7$ and $k = 4$,

$$2746|135 \lessdot_k^q q_{25} 2146|735 \qquad \text{with } q_{25} = q_2 q_3 q_4$$

4,6 ∈ [1,7]

Quantum k-Bruhat order in $S_n[q]$



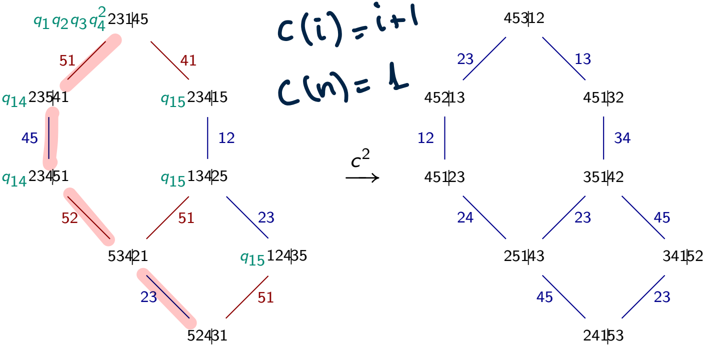


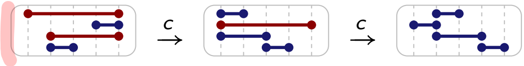
Cyclic symmetry

$$u \lessdot_k^q \mathbf{q}^{\gamma}(\alpha,\beta)u \iff cu \lessdot_k^q \frac{\mathbf{q}^{\gamma}}{\mathbf{q}(u,(\alpha,\beta)u)} c(\alpha,\beta)c^{-1}cu$$

c(i) = i+1

c(n) = 1





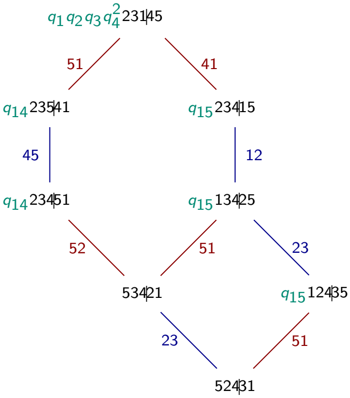
q-HOOK CASE

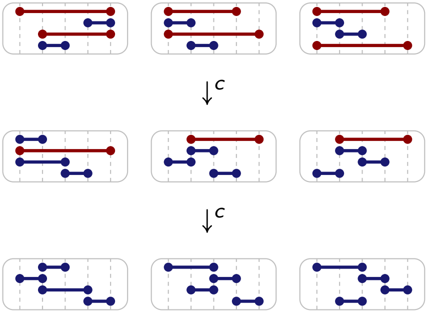
Our first attempt [BBCSS (2022)]]

$$s^{\mathbf{q}}_{\text{hook}(a,b)}(x_1,\dots,x_k)\star\mathfrak{S}^{\mathbf{q}}_u = \sum_{\substack{\gamma:\, u \xrightarrow{\gamma}{}^{\mathbf{q}}_k\, \mathbf{q}^{\alpha(\gamma)} w(\gamma) \\ \exists i: c^i(\gamma) \text{ pure classic} \\ shape(c^{min}(\gamma)) = \text{hook}(a,b)}} \mathbf{q}^{\alpha(\gamma)}\mathfrak{S}^{\mathbf{q}}_{w(\gamma)}$$

- Study some properties of the quantum k-Bruhat poset (equivalent and zero diagrams with 2 and 3 operators).
- Study and understand the terms that do appear in our summation (sequences of operators that cannot happen).
- Understand the cyclic shift transformation for the intervals we have (when they become purely classic).
- Show that those are the terms appearing in the end (including that no other terms can appear).

Example





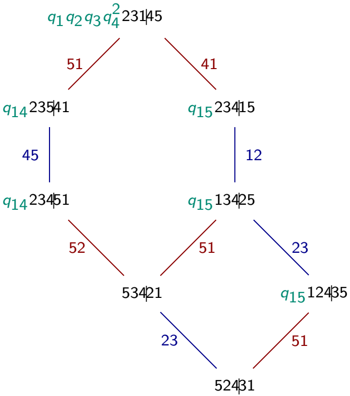
$$s^{\mathbf{q}}_{211}(x_1, x_2, x_3) \star \mathfrak{S}^q_{52431} = \mathbf{1}\, q_1 q_2 q_3 q_4^2 \mathfrak{S}^q_{23145} \quad + \quad \cdots$$

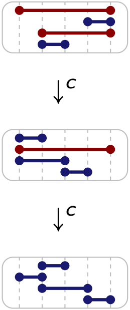
q-Murnaghan–Nakayama's rule

Our (hopefully) theorem

$$p_r^{\mathbf{q}}(x_1,\ldots,x_k) \star \mathfrak{S}_u^{\mathbf{q}} = \sum_{\substack{q^\alpha w:\ \exists\gamma: u \xrightarrow{\gamma}{}_k^{\mathbf{q}} q^\alpha w \\ \exists i: c^i(\gamma) \text{ purely classic, connected} \\ \exists a,b: a+b=r,\ shape(c^{min}(\gamma)) = }} (-1)^{a-1} \mathbf{q}^\alpha \mathfrak{S}_w^{\mathbf{q}}$$

Example





$$p_4^{\mathbf{q}}(x_1, x_2, x_3) \star \mathfrak{S}_{52431}^{\mathbf{q}} = (-\mathbf{1})^{\mathbf{2}}\, q_1 q_2 q_3 q_4^2 \mathfrak{S}_{23145}^{\mathbf{q}} \quad + \quad \cdots$$

In the recent months...

Theorem [BBCSS (2022)]]

Let $u \in S_n$, $l \le k$ and $m \le n - k$. Then

$$\mathfrak{S}_u * s^q_{(m,1^{l-1})}(x_1, \ldots, x_k) = \sum \binom{s(wu^{-1})-1}{\mathrm{ht}(wu^{-1})-l} q^\alpha \mathfrak{S}_w \,,$$

over all minimal intervals $[u, q^\alpha w]^q_k$ s. t. $\ell(q^\alpha w) - \ell(u) = m + l - 1$.

Corollary [BBCSS (2022), conjectured by Morrison (2014)]

Let $u \in S_n$. Then

$$\mathfrak{S}_u * p^q_r(x_1, \ldots, x_k) = \sum (-1)^{\mathrm{ht}(w^{-1}u)+1} q^\alpha \mathfrak{S}_w,$$

over all minimal intervals $[u, q^\alpha w]^q_k$ of rank r s. t. $w^{-1}u$ is a single cycle.

Our proof

$$\mathfrak{S}_u * \mathfrak{S}_{v(\lambda,k)} = \sum C^{q^\alpha w}_{v(\lambda,k),u} \mathfrak{S}_w$$

$$C^{q^\alpha w}_{v(\lambda,k),u} = \binom{s(wu^{-1})-1}{\mathrm{ht}(wu^{-1})-l} q^\alpha \text{ or } 0$$

- Part 1: If $[u, q^\alpha w]^q_k$ is a minimal interval, then there is a hook partition λ with $|\lambda| = \ell(q^\alpha w) - \ell(u)$ such that $C^{q^\alpha w}_{v(\lambda,k),u} \neq 0$.
  - Study of chains in the interval $[u, q^\alpha w]^q_k$ in terms of certain left operators

- Part 2: "quantum equals classical" using a result by Leung–Li (2012) that essentially says
  - If $C^{q^\alpha w}_{v(\lambda,k),u} \neq 0$ for some partition λ, then there are $y, z \in S_n$ with $y \leq_k z$, $zy^{-1} = wu^{-1}$, and the *correct length*, and such that for all partitions μ with $|\mu| = |\lambda|$, $C^{q^\alpha w}_{v(\mu,k),u} = q^\alpha c^z_{v(\mu,k),y}$.
  - If λ is a hook partition, then the interval $[u, q^\alpha w]^q_k$ is minimal.
  - The result follows then by using the classical result.

¡Muchas gracias!

Path $X : [0,1] \to \mathbb{R}^d$ continuous map "smooth enough".
$t \mapsto (X_1(t) \dots X_d(t))$ coordinates

Signature Fix $(i_1 \dots i_k) \in \{1 \dots d\}^k$

$$\sigma_{(i_1 \dots i_k)}(X) := \int_0^1 \int_0^{t_k} \cdots \int_0^{t_2} dX_{i_1}(t_1) \dots dX_{i_k}(t_k) \in \mathbb{R}$$

$$= \int_0^1 \int_0^{t_k} \cdots \int_0^{t_2} \dot{X}_{i_1} \dot{X}_{i_2} \cdots \dot{X}_{i_k} \, dt_1 \dots dt_k$$

$X_{i_2}(t_2)$

Storing these numbers:

k-th signature $\sigma^{(k)}(X) = \left(\sigma_{(i_1 \dots i_k)} \mid i_1 \dots i_k \in \{1 \dots d\} \right)$

$k=2 \rightsquigarrow$ computing areas

$k=3 \rightsquigarrow$ computing volumes?

$k=3, d=2$ 111, 112, 121, 211, ..., 222

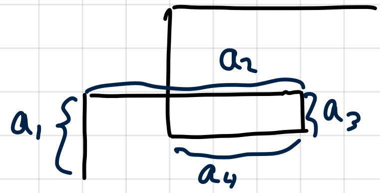
signature of X $\sigma(X) = \left(\sigma^{(k)}(X) \mid k \in \mathbb{N} \right)$, $\sigma^{(0)}(X) = 1$

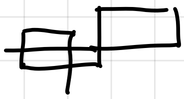
Stochastic analysis

- Introduced by Chen in the 1950s
  $X : [0,T] \to \mathbb{R}^d$

- Up to some "natural relation" the signatures of path determines the path
  issue: $\sigma(X)$ is an infinite sequence.

- Carlos Améndola, Bern Sturmfels & Peter Friz
  $\rightsquigarrow$ varieties of signatures for some path.

$d = 2$ 

axis-parallel paths ⇝ axis paths 

1, 2, 1, 2, 1, 2

$X : [0,1] \longrightarrow \mathbb{R}^d$

$$X(t) = \begin{cases} a_1 e_{v_1} & \text{for } 0 \le t \le T_1 \\ \quad + a_2 e_{v_2} & \text{for } T_1 \le t \le T_2 \\ \quad \vdots & \\ \quad + a_m e_{v_m} & \text{for } T_{m-1} \le t \le T_m \end{cases}$$

$e_{v_1} \dots e_{v_m}$ are standard basis elements in $\mathbb{R}^d$

$X(t)$ has m steps & each step is codified by:

a_i = length of i^{th} step $\longrightarrow$ $\bar{a} = (a_1 \dots a_m)$

$v_i \in \{1 \dots d\}$ directions

↳ $(v_1 \dots v_m)$ sequence

↳ set partition: $\Pi = \{\Pi_1 | \Pi_2 \dots | \Pi_d\}$

$$\Pi_i = \{ j \mid v_j = i \}$$

$$\overset{\;\,1\;\;2\;\;3\;\;4\;\;5\;\;6\;\;7\;\;8}{v = (1,2,1,3,2,3,1,4)} \rightsquigarrow \Pi_v = \{1,3,7 | 2,5 | 4,6 | 8\}$$

signature

$$\sigma_{(i_1 \dots i_k)}(X) = \sum_{(j_1 \dots j_k)} \frac{1}{s_1! \dots s_d!} a_{j_1} \dots a_{j_k}$$ where the sum

runs over all sequences $(j_1 \dots j_k)$ s.t. $1 \le j_1 \le \dots \le j_k \le m$ & $j_\ell \in \Pi_{i_\ell}$ and $s_i = \# \{ j_\ell = i \}$

$v = (1,2,1,3,2,3,1,4)$, $\Pi = \{137 | 25 | 46 | 8\}$ $\bar{a} = (a_1 \dots a_8)$

$$\sigma_{1234} = a_1 a_2 a_4 a_8 + a_1 a_2 a_6 a_8 + a_1 a_5 a_6 a_8 + a_3 a_5 a_6 a_8$$

$\sigma_{2314} = a_2 a_4 a_7 a_8 + a_2 a_6 a_7 a_8 + a_5 a_6 a_7 a_8$

$\sigma_{4123} = 0$ $\sigma_{1122} = \frac{1}{2!} a_1^2 a_2 a_5 + \frac{1}{2!2!} a_1^2 a_2^2 + \dots$

Algebraic geometric (general)

In general, $f: \mathbb{R}^p \to \mathbb{R}^q$ arbitrary map where the coordinates are homogeneous polynomials in p variables & degree k

$\rightsquigarrow \operatorname{im}(f)$

$$f^*: \mathbb{R}[x_1 \dots x_q] \longrightarrow \mathbb{R}[x_1 \dots x_p]$$
$$x_i \longmapsto f_i(x_1 \dots x_p) \quad \forall i = 1 \dots q$$

Im(f) is represented by an ideal $F = $ kernel (f^*).

↳ semialgebraic set in $\mathbb{R}^q \rightsquigarrow$ complicated

① Central — Complex
③ City — Closure (Zariski)
② Police — Projective space
Department — Do whatever you need to do.

① $\operatorname{im}(f) = \mathcal{F}^{im}$

$V^{\mathbb{C}}(F) = \mathcal{F}^{\mathbb{C}} \subseteq \mathbb{C}^q$

$V^{\mathbb{R}}(F) = \mathcal{F}^{\mathbb{R}} \subseteq \mathbb{R}^q$

$\mathcal{F}^{im} \subseteq \mathcal{F}^{\mathbb{R}} \subseteq \mathcal{F}^{\mathbb{C}}$

② Geometrically $\rightsquigarrow$ projective

$f: \mathbb{R}^p \to \mathbb{R}^q$ $\overset{\text{induces}}{\rightsquigarrow}$ $f': \mathbb{P}_{\mathbb{C}}^{p-1} \longrightarrow \mathbb{P}_{\mathbb{C}}^{q-1}$ rational map

$\downarrow$ restrict

$f'': \mathbb{P}_{\mathbb{R}}^{p-1} \longrightarrow \mathbb{P}_{\mathbb{R}}^{q-1}$ rational map.

③ Zariski closure: smallest variety containing a set.

Rk: Computations are done in $\mathbb{Q}$.

$\sigma^{(k)}(X): \mathbb{R}^m \longrightarrow \mathbb{R}^N \qquad N = d^k$

$(a_1 \dots a_m) \longmapsto (\sigma_{i_1 \dots i_k})$

$(\sigma^{(k)})^*: \mathbb{R}[X_1 \dots X_N] \longrightarrow \mathbb{R}[a_1 \dots a_m]$

$X_i = \sigma_{i_1 \dots i_k} \longmapsto \sigma_{i_1 \dots i_k}(a_1 \dots a_m)$

lexicograph order ↳ formula for the signature that we saw before

Variety: $\mathcal{A}_{\nu,k}$ k: level of the signature

ν: sequence of directions

axis path variety (parallel)

Assumptions: • ν doesn't have consecutive repeated entries

• $d = \max(\nu_i)$

[AFS] ① Universal variety for smooth path ($N = d^k$)

$\mathcal{U}_{d,k}$: smallest projective variety in $\mathbb{P}^{N-1}$ that contains $\sigma^{(k)}(X)$ for all X smooth paths in $\mathbb{R}^d$

② $\mathcal{L}_{d,k,m}$: same idea for piecewise linear paths with m steps in $\mathbb{R}^d$, $\sigma^{(k)}(X)$.

- $\mathcal{A}_{\nu,k} \subset \mathcal{U}_{d,k}$ $\quad d \geq \max(\nu_i)$
- $\mathcal{L}_{d,k,m} = \mathcal{U}_{d,k}$ for $m \gg 0$

  Every smooth path can be approximated by using "enough" piecewise linear paths.

- $\forall d,k \ \exists \nu$ s.t. $\max(\nu_i) = d$ & $\mathcal{A}_{\nu,k} = \mathcal{U}_{d,k}$

Q: What did we do?

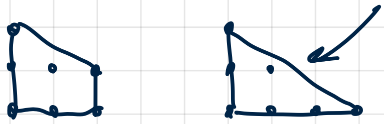
A: Dimensions & toricness & some other properties.

$\mathcal{A}_{(1,2,1),3}$ is toric $\rightsquigarrow$ degree 6 surface in $\mathbb{P}^6$

2-dim lattice polytope of normalize area 6 & that contains exactly 7 points.

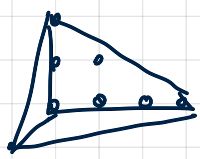
Check Hilbert & Erhart polynomials & Betti table

$\rightarrow$ 2 candidates



Looking at intersections of the tangent spaces at singular points

For $\mathcal{U}_{3,3}$



• Determinants of signatures

$$\left|\sigma^{(2)}(X)\right|_{d=3} = \det\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = P(a)^2 \quad \text{where}$$

$$P(a) = \sum_{\mu=(v_{i_1}\ldots v_{id})} (\text{sgn } \mu) \prod_{j=1}^{d} a_{i_j}$$

"good shapes"

{ F. Galuppi
{ M. Michałek

Second proj: w/ Rosa Preiß

<u>RQ</u>: X path in $\mathbb{R}^d \rightsquigarrow \sigma(X)$

$p: \mathbb{R}^d \to \mathbb{R}^m \longrightarrow p(X)$ "image path"

polynomial map $\quad$ $\S$

$p(0)=0 \quad \sigma(p(X))$

How are $\sigma(X)$ & $\sigma(p(X))$ related?

<u>Q</u>: $\sigma(X) \rightsquigarrow$ Formal power series

$$\sigma(X) = \left(\sigma^{(k)}(X) \mid k \in \mathbb{N}\right)$$

$\sigma_{i_1\ldots i_k}(X) \cdot i_1 \ldots i_k \rightsquigarrow \sigma^{(k)}(X) = \sum_{(i_1,\ldots,i_k)} \sigma_{i_1,\ldots,i_k}(X) \cdot i_1 \ldots i_k$

$\sigma_{i_1\ldots i_k} \in \mathbb{R}$; $i_1\ldots i_k$: a word of length k in $\{1\ldots d\}$

polynomial in the "words"

$\sigma(X) \rightsquigarrow$ Formal power series in the set of words in $\{1\ldots d\}$ $= T((\mathbb{R}^d))$

Algebraic dual: tensor algebra $T(\mathbb{R}^d)$

$(T((\mathbb{R}^d)), \cdot)$ $\quad u \cdot v = uv$ concatenation $\quad 12 \cdot 34 = 1234$

↳ non-commutative algebra

$(T(\mathbb{R}^d), ш)$ $\quad u ш v =$ "interleaves two products in all order-preserving ways"

$$12 ш 34 = 1234 + 1324 + 1342 + 3124 + 3142 + 3412$$

Thm ①

$$\begin{array}{ccc} \mathbb{R}[X_1 \dots X_m] & \xrightarrow{p^*} & \mathbb{R}[X_1 \dots X_d] \\ \downarrow \varphi_m & \circlearrowright & \downarrow \varphi_d \\ \operatorname{im}(\varphi_m) & \xrightarrow{M_p|_{\operatorname{im}(\varphi_m)}} & \operatorname{im}(\varphi_d) \\ \cap & & \cap \\ (T(\mathbb{R}^m), ш) & \xrightarrow[\exists! M_p]{} & (T(\mathbb{R}^d), ш) \end{array}$$

$\varphi_m : X_i \longrightarrow i$

Thm ② $\quad \sigma(p(X)) = M_p^*(\sigma(X))$