Segre zeta functions, log-concavity, and integral dependence

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Segre classes

$Z \hookrightarrow X = \mathbb{P}^n_{\Bbbk}$ be a closed embedding. The Segre class is defined as follows:

$C = C_Z X = \mathcal{S}pec_Z\left(\mathrm{gr}_{\mathscr{I}_Z}(\mathcal{O}_X)\right)$ normal cone, where $\mathrm{gr}_{\mathscr{I}_Z}(\mathcal{O}_X) = \bigoplus_{k \geq 0} \mathscr{I}_Z^k / \mathscr{I}_Z^{k+1}$

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Why do we care about these classes?

- Encode information about the embedding $Z \hookrightarrow X$.
- Play a central role in Fulton-MacPherson's Intersection Theory.

A defining property

$Z \hookrightarrow \mathbb{P}^n_{\Bbbk}$ a closed embedding.

- If Z is regularly embedded in $\mathbb{P}^n_{\Bbbk}$ with normal bundle $N_Z\mathbb{P}^n_{\Bbbk}$, then

$$s(Z, \mathbb{P}^n_{\Bbbk}) \;=\; c\left(N_Z\mathbb{P}^n_{\Bbbk}\right)^{-1} \frown [Z]\,.$$

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- Segre classes have a **birational invariance** property.

- We have the formula

$$s(Z, \mathbb{P}^n_{\mathbb{k}}) \;=\; \eta_*\left(s(E, \mathcal{B})\right) \;=\; \eta_*\left(c\left(N_E\mathcal{B}\right)^{-1} \frown [E]\right) \;=\; \eta_*\left(\frac{[E]}{1+E}\right) \;\in\; A_*(Z),$$

where $\mathcal{B} = \mathcal{B}\ell_Z(\mathbb{P}^n_{\mathbb{k}})$ is the blow-up and $\eta : E = E_Z\mathbb{P}^n_{\mathbb{k}} \to Z$ is the exceptional divisor.

Aluffi's Segre zeta function

Let $I \subset R = \Bbbk[x_0, \ldots, x_n]$ be a homogeneous ideal and $Z = V(I) \subset \mathbb{P}^n_{\Bbbk}$. Write

$$\iota_* \left(s(Z, \mathbb{P}^n_{\Bbbk})\right) \;=\; (a_0 + a_1 H + \cdots + a_n H^n) \frown [\mathbb{P}^n_{\Bbbk}] \;\; \in \;\; A^*(\mathbb{P}^n_{\Bbbk}) \cong \mathbb{Z}[H]/(H^{n+1}).$$

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For all $N \geq n$, let $I^N := I \cdot \Bbbk[x_0, \ldots, x_n, x_{n+1}, \ldots, x_N]$ and $Z^N := V(I^N) \subset \mathbb{P}^N_\Bbbk$, and write

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Definition (Aluffi, 2017)

The **Segre zeta function** of I is given by

$$\zeta_I(t) \;:=\; \sum_{i \geq 0} a_i t^i \;\in\; \mathbb{Z}[[t]].$$

Theorem (Aluffi, 2017) [Rationality]

Let $I \subset R = \Bbbk[x_0, \ldots, x_n]$ be a homogeneous ideal. Write $I = (f_1, \ldots, f_s)$. Then

$$\zeta_I(t) \;=\; \frac{P(t)}{(1 + d_1 t) \cdots (1 + d_s t)}$$

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Theorem (Aluffi, 2017) [Only depends on integral dependence]

Let $I \subset R$ be a homogeneous ideal. Then

$$\zeta_I(t) \;=\; \zeta_{\overline{I}}(t).$$

Integral closure:

$$\overline{I} \;:=\; \left\{ f \in R \mid f^m + a_1 f^{m-1} + a_2 f^{m-2} + \cdots + a_m = 0 \text{ such that } a_i \in I^i \right\}.$$

Example

Let $I = (f_1, \ldots, f_s)$ be a complete intersection (i.e., $f_1, \ldots, f_s$ is a regular sequence), with $d_i = \deg(f_i)$, then

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- Z^N is regularly embedded in $\mathbb{P}^N_{\Bbbk}$: $\quad s(Z^N, \mathbb{P}^N_{\Bbbk}) = c(N_{Z^N}\mathbb{P}^N_{\Bbbk})^{-1} \frown [Z^N]$.

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Example

Let $I_1 = (x_0, x_1, \ldots, x_n)$ and $I_2 = (x_0^2, x_1, \ldots, x_n)$. We have $V(I_1) = \varnothing = V(I_2)$. However:

$$\zeta_{I_1}(t) \;=\; \frac{t^{n+1}}{(1+t)^{n+1}} \quad \text{and} \quad \zeta_{I_2}(t) \;=\; \frac{2t^{n+1}}{(1+2t)(1+t)^n}.$$

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- The mixed version of Segre zeta functions exhibits a log-concavity behavior (we get **Lorentzian polynomials**).

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⋆ Segre classes actually determine integral dependence.

Numerical criteria to detect integral dependence

Theorem (Rees, 1961)

Let $(R, \mathfrak{m})$ be a equidimensional and universally catenary Noetherian local ring and $I \subseteq J$ be two $\mathfrak{m}$-primary ideals in R. Then $\overline{I} = \overline{J}$ if and only if $e(I) = e(J)$.

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Tons of research after Rees' theorem (singularity theory and commutative algebra):

- Teissier's Principle of Specialization of Integral dependence.
- j-multiplicity (Achilles–Manaresi).
- ϵ-multiplicity (Kleiman–Ulrich–Validashti).
- (mixed) Buchsbaum-Rim multiplicity (Kleiman–Thorup).
- Gaffney–Gassler.
- Polini–Trung–Ulrich–Validashti.

Rees' theorem for monomial ideals

Let $I \subset \Bbbk[x_1, \ldots, x_d]$ be a **monomial ideal**.

Lemma: We have $\overline{I} = \left(\mathbf{x}^{\mathbf{n}} \mid \mathbf{n} \in \mathrm{NP}(I) \cap \mathbb{N}^d\right)$ where $\mathrm{NP}(I) := \mathrm{conv}\{\mathbf{n} \mid \mathbf{x}^{\mathbf{n}} \in I\}$ is the Newton polyhedron of I.

Theorem (Teissier, 1988): Assume $\dim(R/I) = 0$.

Then we have $e(I) := \lim_{n\to\infty} \frac{\ell(R/I^n)}{n^d/d!} = d! \cdot \mathrm{vol}\left(\mathbb{R}^d_{\geq 0} \setminus \mathrm{NP}(I)\right)$.

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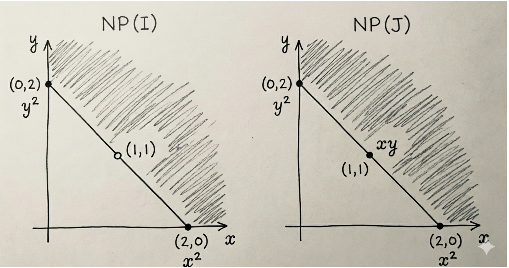
In $\mathbb{k}[x,y]$, we have $I = (x^2, y^2) \subsetneq J = (x^2, xy, y^2)$ and $\overline{I} = \overline{J}$ and $e(I) = e(J) = 4$.



Segre classes and integral dependence

Corollary (Cid-Ruiz, 2025)

Let $I \subseteq J \subset R$ be two homogeneous ideals in $R = \Bbbk[x_0, \ldots, x_n]$. Then $\overline{I} = \overline{J}$ if and only if $\zeta_I(t) = \zeta_J(t)$.

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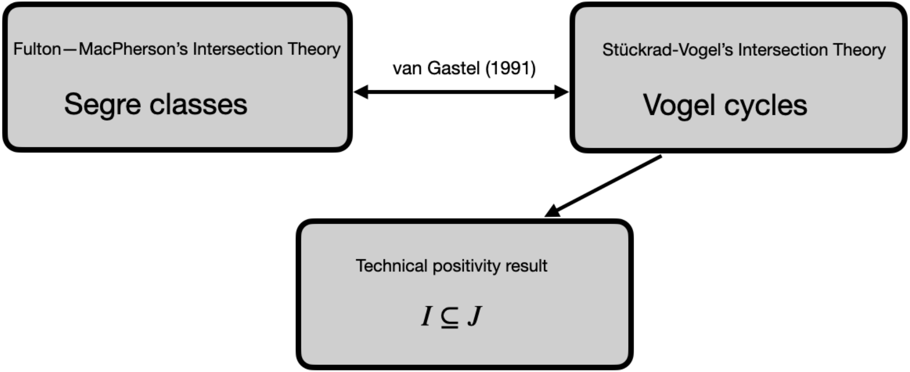
Theorem (Cid-Ruiz, 2025)

Let X be an equidimensional projective scheme over a field $\Bbbk$. Let $\mathscr{L}$ be an ample line bundle on X and $W \subseteq Z$ be two closed subschemes of X. Then the following three conditions are equivalent:

a. $\overline{\mathscr{I}_Z} = \overline{\mathscr{I}_W}$.
b. $s(Z, X) = s(W, X)$ viewed in $A_*(X)$.
c. $\deg_{\mathscr{L}}\left(s^i(Z, X)\right) = \deg_{\mathscr{L}}\left(s^i(W, X)\right)$ for all $i \geq 0$.

General idea



Refined Bézout theorem (Stückrad-Vogel) [for improper intersections]

A. THE STATEMENT OF THE MAIN THEOREM.

(2.1) MAIN THEOREM. Let $V_1 = V(I_1)$ and $V_2 = V(I_2)$ be two pure dimensional subvarieties in $\mathbb{P}^n_K$ defined by homogeneous ideals I_1 and I_2 in $K[X_0,\dots,X_n]$. There exists a collection $\{C_i\}$ of irreducible subvarieties of $V_1 \cap V_2$ (one of which may be ϕ) such that

(i) For every $C_i \in \{C_i\}$ there are intersection numbers, say $j(V_1,V_2;C_i) \geq 1$ of V_1 and V_2 along C_i given by the lengths of certain well-defined primary ideals such that

$$\deg(V_1)\cdot\deg(V_2) = \sum_{C_i \in \{C_i\}} j(V_1,V_2;C_i)\cdot\deg(C_i),$$

where we put $\deg(\phi) = 1$.

(ii) If $C \subset V_1 \cap V_2$ is an irreducible component of $V_1 \cap V_2$ then $C_i \in \{C_i\}$.

(iii) For every $C_i \in \{C_i\}$

$$\dim(C_i) \geq \dim(V_1) + \dim(V_2) - n.$$

Historical context

Commutative Algebra

- Rees
- Kirby
- Northcott
- Lipman—Sataye
- Hochster—Huneke

Singularity Theory

- Teissier
- Briançon
- Skoda
- Merle
- Gaffney

Kleiman—Thorup

- **A Geometric Theory of the Buchsbaum—Rim Multiplicity** (1994)
- **Mixed Buchsbaum-Rim multiplicities** (1996)

Introduced mixed Segre classes

Mixed Segre classes (Kleiman–Thorup)

"Definition" by exploiting birational invariance:

- $Z_1, \ldots, Z_m \subset \mathbb{P}^n_{\Bbbk}$, where $Z_i = V(I_i)$ and $I_i \subset R = \Bbbk[x_0, \ldots, x_n]$.

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Definition

For each $\mathbf{i} = i_1, \ldots, i_m \in \mathbb{N}$, the **i-th mixed Segre class** is given by

$$s^{i_1,\ldots,i_m}(Z_1, \ldots, Z_m; \mathbb{P}^n_{\Bbbk}) := (-1)^{i_1+\cdots+i_m-1}\eta_*\left(E_1^{i_1}\cdots E_m^{i_m}\right) \in A_*(Z).$$

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The **total mixed Segre class** is given by

$$s(Z_1, \ldots, Z_m; \mathbb{P}^n_{\mathbb{k}}) := \sum_{i_1,\ldots,i_m\in\mathbb{N}} s^{i_1,\ldots,i_m}(Z_1, \ldots, Z_m; \mathbb{P}^n_{\mathbb{k}})\, t_1^{i_1}\cdots t_m^{i_m} \in A_*(Z)[t_1, \ldots, t_m].$$

$Z_1, \ldots, Z_m \subset \mathbb{P}^n_{\Bbbk}$, where $Z_i = V(I_i)$ and $I_i \subset R = \Bbbk[x_0, \ldots, x_n]$.

Theorem (Kleiman–Thorup) [Mixed formula]

For all $r \geq 0$, we have

$$s^r(Z, \mathbb{P}^n_{\Bbbk}) \;=\; \sum_{i_1+\cdots+i_m=r} \frac{r!}{i_1! \cdots i_m!} s^{i_1,\ldots,i_m}(Z_1, \ldots, Z_m; \mathbb{P}^n_{\Bbbk}) \;\in\; A_*(Z).$$

$Z_1, \ldots, Z_m \subset \mathbb{P}^n_{\Bbbk}$, where $Z_i = V(I_i)$ and $I_i \subset R = \Bbbk[x_0, \ldots, x_n]$.

Theorem (Kleiman–Thorup) [Mixed formula]

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Theorem (Kleiman–Thorup) [Blow-up formula]

$$\iota_* \left(s(Z_1, \ldots, Z_m; \mathbb{P}^n_{\Bbbk})\right) \; = \; [\mathbb{P}^n_{\Bbbk}] - \pi_* \left(\frac{1}{(1+E_1 t_1) \cdots (1+E_m t_m)} \frown [\mathcal{B}\ell_Z(\mathbb{P}^n_{\Bbbk})] \right).$$

Mixed Segre zeta function

$Z_1, \ldots, Z_m \subset \mathbb{P}^n_{\Bbbk}$, where $Z_i = V(I_i)$ and $I_i \subset R = \Bbbk[x_0, \ldots, x_n]$.

For all $N \geq n$, let $I_i^N := I_i \cdot \Bbbk[x_0, \ldots, x_n, x_{n+1}, \ldots, x_N]$ and $Z_i^N := V(I_i^N) \subset \mathbb{P}^N_{\Bbbk}$, and write

$$\iota_* \left(s^{i_1, \ldots, i_m}(Z_1^N, \ldots, Z_m^N; \mathbb{P}^N_{\Bbbk}) \right) \; = \; a_{i_1, \ldots, i_m} H^{i_1 + \cdots + i_m} \frown \left[\mathbb{P}^N_{\Bbbk} \right] \; \in \; A_{N-(i_1+\cdots+i_m)}(\mathbb{P}^N_{\Bbbk}).$$

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Definition

The **mixed Segre zeta function** of $I_1, \ldots, I_m$ is given by

$$\zeta_{I_1, \ldots, I_m}(t_1, \ldots, t_m) \;:=\; \sum_{i_1, \ldots, i_m \in \mathbb{N}} a_{i_1, \ldots, i_m} t_1^{i_1} \cdots t_m^{i_m} \;\in\; \mathbb{Z}[[t_1, \ldots, t_m]].$$

Theorem (Cid-Ruiz, 2025) [Rationality]

Let $I_1, \ldots, I_m \subset R = \Bbbk[x_0, \ldots, x_n]$ be homogeneous ideals. Write $I_i = (f_{i,1}, \ldots, f_{i,s_i})$. Then

$$\zeta_{I_1,\ldots,I_m}(t_1, \ldots, t_m) \;=\; \frac{P(t_1, \ldots, t_m)}{\prod_{i,j}\left(1 + d_{i,j} t_i\right)}$$

with $P(t_1, \ldots, t_m) \in \mathbb{N}[t_1, \ldots, t_m]$, and where $d_{i,j} = \deg(f_{i,j})$.

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Theorem (Cid-Ruiz, 2025) [Only depends on integral dependence]

Let $I_1, \ldots, I_m \subset R$ be homogeneous ideals. Then

$$\zeta_{I_1,\ldots,I_m}(t_1, \ldots, t_m) \;=\; \zeta_{\overline{I_1},\ldots,\overline{I_m}}(t_1, \ldots, t_m).$$

Example

Let $I_1 = (f_1, \ldots, f_r)$ and $I_2 = (g_1, \ldots, g_s)$ be complete intersections in $R = \mathbb{k}[x_0, \ldots, x_n]$. Assume that $f_i \in \mathbb{k}[x_0, \ldots, x_k]$ and $g_i \in \mathbb{k}[x_{k+1}, \ldots, x_n]$. Let $a_i = \deg(f_i)$ and $b_i = \deg(g_i)$. Then

$$\zeta_{I_1,I_2}(t_1, t_2) \;=\; \zeta_{I_1}(t_1) + \zeta_{I_2}(t_2) - \zeta_{I_1}(t_1)\zeta_{I_2}(t_2)$$

where

$$\zeta_{I_1}(t_1) \;=\; \frac{a_1 \cdots a_r t_1^r}{(1 + a_1 t_1) \cdots (1 + a_r t_1)} \quad \text{and} \quad \zeta_{I_2}(t_2) \;=\; \frac{b_1 \cdots b_s t_2^s}{(1 + b_1 t_2) \cdots (1 + b_s t_2)}.$$

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$$\zeta_{I_1}(t_1) \;=\; \frac{a_1 \cdots a_r t_1^r}{(1+a_1t_1)\cdots(1+a_rt_1)} \quad \text{and} \quad \zeta_{I_2}(t_2) \;=\; \frac{b_1 \cdots b_s t_2^s}{(1+b_1t_2)\cdots(1+b_st_2)}.$$

Example (continued)

Let $I = I_1 I_2 \subset R$. Write $A(t) = \zeta_{I_1}(t) - 1 = \sum_{i \geq 0} \alpha_i t^i$ and $B(t) = \zeta_{I_2}(t) - 1 = \sum_{i \geq 0} \beta_i t^i$. Then

$$\zeta_I(t) \;=\; 1 \;-\; A(t) \odot B(t)$$

where $A(t) \odot B(t) \;:=\; \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{k}{i} \alpha_i \beta_{k-i} \right) t^k$ is the *binomial convolution*.

Recap on log-concavity

Theorem (Huh, 2012)

Let $X \subset \mathbb{P}_{\Bbbk}^{m_1} \times_{\Bbbk} \mathbb{P}_{\Bbbk}^{m_2}$ be an irreducible subvariety of dimension d. Write $[X] = \sum_{i=0}^{d} a_i \left[\mathbb{P}_{\Bbbk}^{i} \times_{\Bbbk} \mathbb{P}_{\Bbbk}^{d-i}\right] \in A^*\left(\mathbb{P}_{\Bbbk}^{m_1} \times_{\Bbbk} \mathbb{P}_{\Bbbk}^{m_2}\right)$. Then the sequence

$$\left(a_0, a_1, \ldots, a_d\right), \text{ where } a_i = \deg^{i,d-i}(X),$$

has **no intermediate zeroes** and it is **log-concave** (i.e., $a_{i-1}a_{i+1} \leq a_i^2$).

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Theorem (Brändén–Huh, 2020)

Let X be an irreducible projective variety of dimension d and $H_1, \ldots, H_m$ be nef divisors on X. Then the volume polynomial

$$\mathrm{vol}_X(t_1, \ldots, t_m) := \int \left(H_1 t_1 + \cdots + H_m t_m\right)^d \cap [X] \in \mathbb{Q}[t_1, \ldots, t_m]$$

is a **Lorentzian polynomial**.

Lorentzian polynomials

Definition (Brändén-Huh, 2020)

A homogeneous polynomial $f = \sum_{|\alpha|=d} c_\alpha \, t_1^{\alpha_1} \cdots t_m^{\alpha_m} \in \mathbb{R}_{\geq 0}[t_1, \ldots, t_m]$ of degree d is **Lorentzian** if

1. the support is M-convex (equivalently, forms a discrete polymatroid) and
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Remark

When a polynomial $f(t_1, \ldots, t_m) = \sum_{\alpha \in \mathbb{N}^m} c_\alpha t_1^{\alpha_1} \cdots t_m^{\alpha_m}$ is Lorentzian, we have

- $c_\alpha^2 \geq c_{\alpha+\mathbf{e}_i-\mathbf{e}_j} c_{\alpha-\mathbf{e}_i+\mathbf{e}_j}$ (discretely log-concave in all the directions).

Remark (normalization operator)

The linear operator $N(t^\alpha) = t^\alpha/\alpha!$ preserves Lorentzian polynomials. A polynomial f is **denormalized Lorentzian** if $N(f)$ is Lorentzian.

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Theorem (Cid-Ruiz, 2025) [Log-concavity of mixed Segre zeta functions]

Let $I_1, \ldots, I_m \subset R = \Bbbk[x_0, \ldots, x_n]$ be homogeneous ideals. Write $I_i = (f_{i,1}, \ldots, f_{i,s_i})$. Define $Q(t_1, \ldots, t_m)$ by the identity

$$1 - \zeta_{I_1,\ldots,I_m}(t_1, \ldots, t_m) \;=\; \frac{Q(t_1, \ldots, t_m)}{\prod_{i,j}\left(1 + d_{i,j}t_i\right)}.$$

Then the homogenization of $Q(t_1, \ldots, t_m)$ is a denormalized Lorentzian polynomial.

$\underline{\mathbf{Z}} = Z_1, \ldots, Z_m \subset \mathbb{P}^n_{\Bbbk}$, where $Z_i = V(I_i)$ and $I_i \subset R = \Bbbk[x_0, \ldots, x_n]$.
Let $\iota : Z = V(I_1 \cdots I_m) \hookrightarrow \mathbb{P}^n_{\Bbbk}$ and $\pi : \mathcal{B} = \mathcal{B}\ell_Z(\mathbb{P}^n_{\Bbbk}) \to \mathbb{P}^n_{\Bbbk}$.

General idea:

- By the blow-up formula, $1 - \zeta_{I_1,\ldots,I_m}(t_1, \ldots, t_m)$ corresponds with

$$[\mathbb{P}^n_{\Bbbk}] - \iota_* \left(s(\underline{\mathbf{Z}}; \mathbb{P}^n_{\Bbbk})\right) \; = \; \pi_* \left(\frac{1}{(1 + E_1 t_1) \cdots (1 + E_m t_m)} \frown [\mathcal{B}] \right).$$

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- The polynomial $Q(t_1, \ldots, t_m) = \prod_{i,j} \left(1 + d_{i,j} t_i\right) \cdot (1 - \zeta_{I_1,\ldots,I_m}(t_1, \ldots, t_m))$ corresponds with

$$\prod_{i=1}^{m} c_{t_i}\left(\mathscr{E}_i\right) \frown \pi_* \left(\frac{1}{\prod_{i=1}^m c_{t_i}(\mathcal{O}_{\mathcal{B}}(-\mathbf{e}_i))} \frown [\mathcal{B}] \right) \;=\; \pi_* \left(\prod_{i=1}^{m} c_{t_i}(\mathcal{Q}_i) \frown [\mathcal{B}] \right)$$

where $\mathscr{E}_i = \bigoplus_j \mathcal{O}_{\mathbb{P}^n_{\Bbbk}}(d_{i,j})$ and $\mathcal{Q}_i = \pi_*\left(\mathscr{E}_i\right)/\mathcal{O}_{\mathcal{B}}(-\mathbf{e}_i)$.

Thank you!

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