Enumerating Higher Bruhat Orders

Through Deletion and Contraction

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Motivation

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- Topology of discriminantal arrangements (Manin, Schechtman 1989)

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- **Rhombic tilings and Bott-Samelson varieties (Elnitsky 1997) (Escobar, Pechenik, Tenner, Yong 2018)**

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- Cup-i coproducts defining Steenrod squares in cohomology (Laplante-Anfossi, Williams 2023)

Background

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Example ($n = 4$)

$$w_0 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$
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$$= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1.$$

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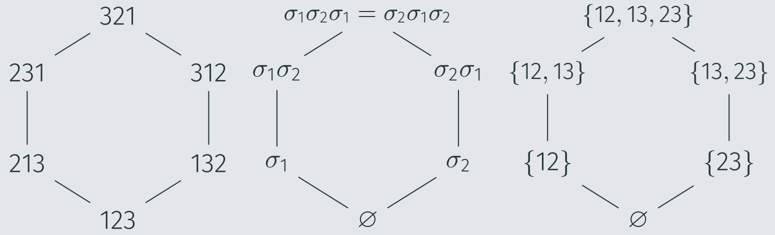
&= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1.
\end{aligned}
$$

Weak Bruhat Order on $\mathfrak{S}_n$

The (right) weak Bruhat order on $\mathfrak{S}_n$ is a partial order where $v \leq w$ if *some reduced word for v is a prefix of some reduced word for w.*

Equivalently, the weak Bruhat order is the transitive closure of cover relations $v \lessdot w$ if and only if $\mathsf{Inv}(w) = \mathsf{Inv}(v) \cup \{(i,j)\}$.

Example ($n = 3$)



Packets

Definition (Manin-Schechtman)

The packet of $X = (x_1, \ldots, x_k) \in \binom{[n]}{k}$ is $P(X) = \{X_k, X_{k-1}, \ldots, X_1\}$ where $X_i = X \setminus \{x_i\}$.

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Two sets $X, Y \in \binom{[n]}{k}$ commute if $P(X) \cap P(Y) = \varnothing$.

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Example ($n = 5, k = 3$)

Let $X = (1, 2, 3)$, $Y = (1, 3, 4)$, $Z = (3, 4, 5)$. Then

$$P(X) = \{12, 13, 23\},$$
$$P(Y) = \{13, 14, 34\},$$
$$P(Z) = \{34, 35, 45\}.$$

X and Y do not commute. Y and Z do not commute. But X and Z commute.

Admissible Orders

Definition (Manin-Schechtman)

A total order ρ on $\binom{[n]}{k}$ is admissible if $\rho|_{P(X)}$ is the lex or antilex order for every $X \in \binom{[n]}{k+1}$.

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Definition (Manin-Schechtman)

The reversal set of $\rho \in \mathcal{A}(n,k)$ is

$$\mathrm{Rev}(\rho) := \{X : \rho|_{P(X)} \text{ is antilex order}\}.$$

Admissible Orders

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Example ($n = 4, k = 2$)

$$\rho^{(1)} = (12, 13, 14, 23, 24, 34),$$
$$\rho^{(2)} = (12, 13, 23, 14, 24, 34),$$
$$\rho^{(3)} = (12, 13, 14, 34, 24, 23).$$

$\rho^{(1)}$ and $\rho^{(2)}$ differ by commuting 14 and 23, and $[\rho^{(1)}] = \{\rho^{(1)}, \rho^{(2)}\}$.

$\rho^{(1)}$ and $\rho^{(3)}$ differ by flipping the packet $P(234)$ from lex to antilex.

Consistent Sets

Definition (Ziegler)

A subset $I \subseteq \binom{[n]}{k}$ is consistent if $I \cap P(X)$ is a prefix or suffix of $P(X)$ for every $X \in \binom{[n]}{k+1}$.

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Lemma (Ziegler)

For every $\rho \in \mathcal{A}(n, k)$, the reversal set $\mathrm{Rev}(\rho) \in \binom{[n]}{k+1}$ is consistent.

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Lemma (Ziegler)

For every $\rho \in \mathcal{A}(n,k)$, the reversal set $\mathrm{Rev}(\rho) \in \binom{[n]}{k+1}$ is consistent.

Example

$\{123, 124\}$ and $\{124, 134, 234\}$ are consistent.

$\mathrm{Rev}(12, 13, 14, 34, 24, 23) = \{234\}$ is consistent.

Higher Bruhat Orders

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Definition (Ziegler)

The higher Bruhat order $\mathcal{C}(n,k+1)$ is the consistent subsets of $\binom{[n]}{k+1}$ with partial order induced by $I \lessdot J$ if and only if $J = I \cup \{X\}$ for some $X \in \binom{[n]}{k+1}$.

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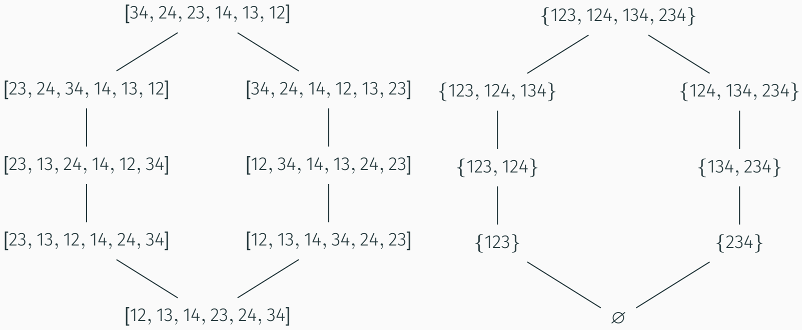
Theorem (Ziegler)

The map $[\rho] \mapsto \mathrm{Rev}(\rho)$ is a poset isomorphism between $\mathcal{B}(n,k)$ and $\mathcal{C}(n,k+1)$.

Higher Bruhat Orders



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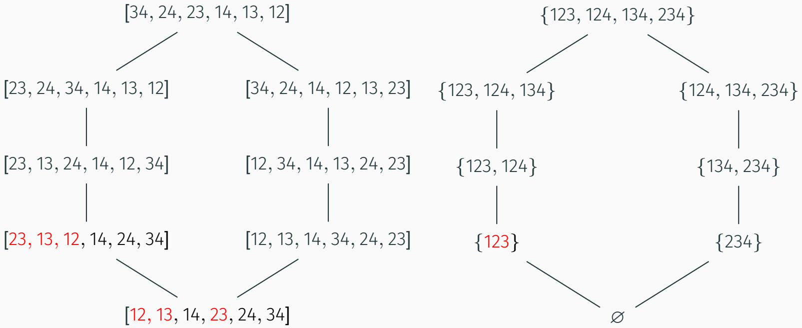

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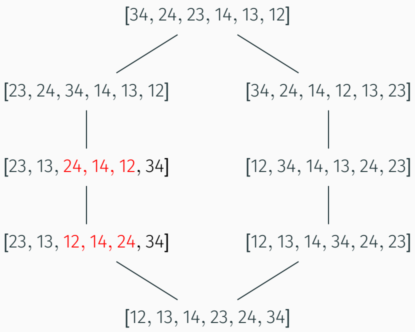

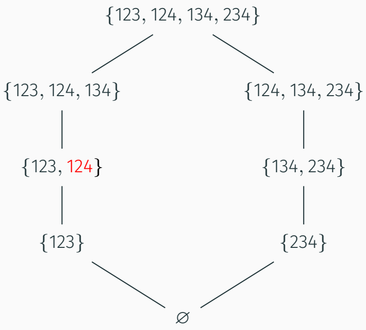

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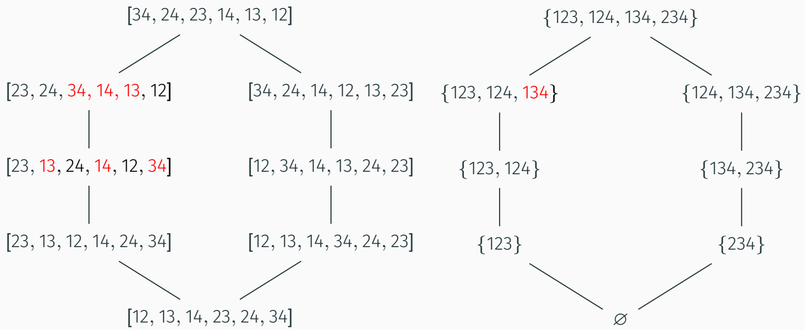

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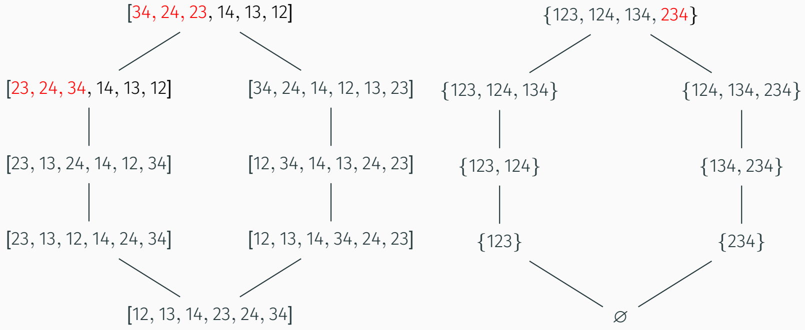

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Fact

The weak Bruhat order on $\mathfrak{S}_n$ is isomorphic to $\mathcal{B}(n, 1) \cong \mathcal{C}(n, 2)$. Admissible orders on $\binom{[n]}{1}$ correspond to permutations and reversal sets correspond to inversions.

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$|\mathcal{A}(n, k)|$ is equal to the number of standard Young tableaux of shape $(n-1, n-2, \ldots, 1)$.

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Question: Is there a closed formula for $|\mathcal{B}(n,2)|$?

This is still **open**!

Known Results

Theorem (Ziegler, 1993)

For $n \geq 4$ we have

$$
\begin{aligned}
|\mathcal{B}(n,n)| &= 1 \\
|\mathcal{B}(n,n-1)| &= 2 \\
|\mathcal{B}(n,n-2)| &= 2n \\
|\mathcal{B}(n,n-3)| &= 2^n + n2^{n-2} - 2n
\end{aligned}
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Theorem (Balko, 2019)

For $k \geq 2$ and sufficiently large $n \gg k$, we have

$$\frac{n^k}{(k+1)^{4(k+1)}} \leq \log_2 |\mathcal{B}(n,k)| \leq \frac{2^{k-1} n^k}{k!}.$$

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Theorem

For sufficiently large k and $n \gg k$, we have

$$\frac{n^k}{\sqrt{24\pi k}\,(k+1)!} \leq \log_2 |\mathcal{B}(n,k)| \leq \frac{n^k}{k! \log 2}.$$

Deletion, Contraction, and Weaving Functions

Deletion and Contraction

Definition

For $I \subseteq \binom{[n]}{k}$, the deletion of I is

$$I \setminus n = I \cap \binom{[n-1]}{k}.$$

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For $I \subseteq \binom{[n]}{k}$, the contraction of I is

$$I/n = \{X \setminus \{n\} : X \in I \text{ and } n \in X\}.$$

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Definition

For a total order ρ on $\binom{[n]}{k}$, the contraction ρ/n is the total order on $\binom{[n-1]}{k-1}$ where $X < Y$ if and only if $X \cup \{n\} < Y \cup \{n\}$ in ρ.

Deletion and Contraction

Example

Let $\rho = (23, 13, 24, 14, 12, 34) \in \mathcal{A}(4, 2)$ and $I = \mathrm{Rev}(\rho) = \{123, 124\}$. Then

$$\rho \setminus 4 = (23, 13, 12)$$
$$\rho/4 = (2, 1, 3)$$
$$I \setminus 4 = \{123\}$$
$$I/4 = \{12\}$$

Deletion and Contraction

Lemma

For all $\rho \in \mathcal{A}(n,k)$ we have $\rho \setminus n \in \mathcal{A}(n-1,k)$ and $\rho/n \in \mathcal{A}(n-1,k-1)$. Furthermore,

$$\mathrm{Rev}(\rho \setminus n) = \mathrm{Rev}(\rho) \setminus n, \text{ and}$$
$$\mathrm{Rev}(\rho/n) = \mathrm{Rev}(\rho)/n, \text{ and}$$

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Lemma

For all $I \in \mathcal{C}(n,k)$ we have $I \setminus n \in \mathcal{C}(n-1,k)$ and $I/n \in \mathcal{C}(n-1,k-1)$.

Deletion and Contraction

Lemma

The map from $\mathcal{C}(n,k)$ to $\mathcal{C}(n-1,k) \times \mathcal{C}(n-1,k-1)$ that sends I to $(I \setminus n, I/n)$ is injective.

Proof.

A consistent set $I \in \mathcal{C}(n,k)$ can be recovered from $(I \setminus n, I/n)$ by the equation

$$I = (I \setminus n) \cup \{X \cup \{n\} : X \in I/n\}.$$

□

Frame Title

Weaving Functions

Example

Let $\rho = (23, 24, 25, 45, 13, 15, 35, 14, 34, 12) \in \mathcal{A}(5, 2)$. Then

$$W_\rho(1) = 0000$$
$$W_\rho(2) = 0001$$
$$W_\rho(3) = 0011$$
$$W_\rho(4) = 1011$$
$$W_\rho(5) = 1111.$$

Weaving Functions

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Theorem

For integers $1 \leq k \leq n$ and $[\rho], [\sigma] \in \mathcal{B}(n, k)$, $[\rho] = [\sigma]$ if and only if $W_\rho = W_\sigma$.

Proof of Theorem

Theorem

For sufficiently large k and $n \gg k$, we have

$$\frac{n^k}{\sqrt{24\pi k}\,(k+1)!} \leq \log_2 |\mathcal{B}(n,k)| \leq \frac{n^k}{k! \log 2}.$$

Proof of Upper Bound

Induction on k with base case $k = 2$.

$$\begin{aligned} \log_2 |\mathcal{B}(n,2)| &\leq \log_2 \prod_{i=1}^{n} \binom{n-1}{i-1} \\ &\leq \frac{n^2}{2\log 2} + O(n \log n) \end{aligned}$$

Proof of Theorem

Proof of Upper Bound

Inductive step. Suppose $\log_2|\mathcal{B}(n,k-1)| \le \frac{n^{k-1}}{(k-1)!\log 2}$. Then

$$\begin{aligned}
|\mathcal{C}(n,k+1)| &\le |\mathcal{C}(n-1,k+1)|\cdot|\mathcal{C}(n-1,k)| \\
&\le |\mathcal{C}(n-2,k+1)|\cdot|\mathcal{C}(n-2,k)|\cdot|\mathcal{C}(n-1,k)| \\
&\vdots \\
&\le \prod_{m=k}^{n-1}|\mathcal{C}(m,k)|.
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&\vdots \\
&\leq \prod_{m=k}^{n-1} |\mathcal{C}(m, k)|.
\end{aligned}$$

Thus $|\mathcal{B}(n, k)| \leq \prod_{m=k}^{n-1} |\mathcal{B}(m, k-1)|$.

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$$|\mathcal{B}(n, k)| \leq \prod_{m=k}^{n-1} |\mathcal{B}(m, k-1)|.$$

Taking logs gives

$$\log_2 |\mathcal{B}(n, k)| \leq \sum_{m=k}^{n-1} \frac{m^{(k-1)}}{(k-1)! \log 2}$$
$$\leq \frac{n^k}{k! \log 2}.$$

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Definition

A subset $I \subset \binom{[n]}{k}$ is coconsistent if $P^*(X) \cap I$ is a prefix or suffix of X in lex order for all $X \in \binom{[n]}{k-1}$.

Duality

Definition

The dual higher Bruhat order $\mathcal{B}^*(n,k)$ is $\mathcal{A}^*(n,k)/\sim$ with partial order induced by $[\rho] \lessdot [\sigma]$ if and only if some $\rho' \in [\rho]$ and $\sigma' \in [\sigma]$ differ by a **copacket** flip from lex to antilex.

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The dual higher Bruhat order $\mathcal{C}^*(n,k)$ is the **coconsistent** subsets of $\binom{[n]}{k+1}$ with partial order induced by $I \lessdot J$ if and only if $J = I \cup \{X\}$ for some $X \in \binom{[n]}{k}$.

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Theorem

The map $[\rho] \mapsto \mathrm{Corev}(\rho)$ is a poset isomorphism between $\mathcal{B}^*(n,k)$ and $\mathcal{C}^*(n,k-1)$.

Duality

Theorem

The following diagram of poset isomorphisms commutes.

$$\begin{array}{ccc} \mathcal{B}(n,k) & \xrightarrow{\text{Rev}} & \mathcal{C}(n,k+1) \\ \downarrow \beta & & \downarrow \gamma \\ \mathcal{B}^*(n,n-k) & \xrightarrow{\text{Corev}} & \mathcal{C}^*(n,n-k-1), \end{array}$$

where β maps

$$(\rho_1,\ldots,\rho_\ell) \mapsto \left(\binom{[n]}{k} \setminus \rho_1, \ldots, \binom{[n]}{k} \setminus \rho_\ell\right)$$

and γ maps

$$I \mapsto \{[n] \setminus X : X \in I\}.$$

Duality

Theorem

For sufficiently large k and $n \gg k$, we have

$$\frac{n^k}{\sqrt{24\pi k}(k+1)!} \leq \log_2 |\mathcal{B}(n,k)| \leq \frac{n^k}{k! \log 2}.$$

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Theorem

For sufficiently large k and $n \gg k$, we have

$$\frac{n^{k-2}}{\sqrt{24\pi(k-2)}\,(k-1)!} \leq \log_2 |\mathcal{B}^*(n,k)| \leq \frac{n^{k-2}}{(k-2)!}.$$

Thank you!