How to prove that $K^{-1}K = I$

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Outline

Set up

- Symmetric functions **Sym**

- K Kostka matrix
  - Counts semistandard Young tableaux
- K^{-1} inverse Kostka matrix
  - Counts special rim hook tableaux (ER '90)
- Showing $\delta_{\lambda,\mu} = (K^{-1}K)_{\lambda,\mu}$ via sign-reversing involution
- Previous work (SL '06)

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- Previous work (SL '06)

Our approach

- Noncommutative symmetric functions **NSym**
- $\tilde{K}$ **NSym** Kostka matrix
  - Counts immaculate tableaux (BBSSZ '14)
- $\tilde{K}^{-1}$ **NSym** inverse Kostka
  - Counts tunnel hook coverings (AM '24)
- Showing $\delta_{\alpha,\beta} = (\tilde{K}^{-1}\tilde{K})_{\alpha,\beta}$ via sign-reversing involution
- Our involution in **NSym**
- Our involution in **Sym**

Symmetric Functions

Formal power series $f \in \mathbb{C}[[x_1, x_2, x_3, \ldots]]$ invariant under any permutation of its indices

$$2x_1^2x_2 + 2x_1x_2^2 + 2x_1^2x_3 + 2x_1x_3^2 + \cdots \qquad \checkmark$$
$$2x_1^2x_2 - 3x_1x_2^2 + 2x_1^2x_3 + 2x_1x_3^2 + \cdots \qquad \text{X}$$

Sym $= \{$symmetric functions$\}$

Set of generators for Sym (complete homogeneous sym. functions)

For $r \in \mathbb{Z}_{>0}$, $h_r = \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r}$ and with $h_0 = 1$.

Sym$_n$ $= \{$symmetric functions of homogeneous degree $n\}$

Basis of Sym$_n$

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash n$ (partition of n), $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$

Schur functions

Semistandard Young tableau (SSYT) of shape λ: a filling of a diagram of λ by positive integers that has

- weakly increasing rows, and
- strictly increasing columns.

1	2	2	3	4
2	3	5	5	6
4	5			
6				

shape=λ=(5,5,2,1)

content=μ=(1,3,2,2,3,2)

$$x^T := \prod_{i \geq 1} x_i^{\#\text{ of } i \text{ in } T}$$

$$= x_1 x_2^3 x_3^2 x_4^2 x_5^3 x_6^2$$

Schur function s_λ of shape λ

$$s_\lambda = \sum_{T \in SSYT_\lambda} x^T$$

Jacobi–Trudi identity

$$s_\lambda = \det(h_{\lambda_i + j - i})_{i,j=1}^{\ell(\lambda)}$$

where $h_{-k} = 0$ for $k > 0$

The Kostka matrix

The (λ, μ) entry of the Kostka matrix K is the Kostka number $K_{\lambda,\mu}$ of SSYT T of shape λ and content μ.

$$K = (K_{\lambda,\mu}) = \begin{array}{c} \\ 4 \\ 31 \\ 22 \\ 211 \\ 1111 \end{array} \begin{array}{c} \begin{array}{ccccc} 4 & 31 & 22 & 211 & 1111 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Proposition (K is the transition matrix between h and s)

For all $\mu \vdash n$,

$$h_\mu = \sum_{\lambda \vdash n} K_{\lambda,\mu} s_\lambda$$

Inverse Kostka matrix

K is upper unitriangular, so it has an inverse.

$$\underset{K}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \qquad \underset{K^{-1}}{\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}$$

Two combinatorial interpretation of K^{-1} (in this talk):

- Special rim hook tableaux (Eğecioğlu–Remmel 1990)
- Tunnel hook coverings (Allen–Mason 2023)

Inverse Kostka matrix

K is upper unitriangular, so it has an inverse.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K \qquad\qquad\qquad K^{-1}$$

Two combinatorial interpretation of K^{-1} (in this talk):

- Special rim hook tableaux (Eğecioğlu–Remmel 1990)
- Tunnel hook coverings (Allen–Mason 2023) $\leftarrow$ use this!

Special rim hook tableaux (Eğecioğlu–Remmel 1990)

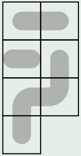
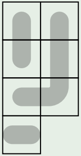
Special rim hook tableaux: covering of the diagram with lattice paths with a cell in the leftmost column

$\mathrm{SRT}_{\lambda,\mu} = \{$ special rim hook tableaux with content λ and shape $\mu\}$

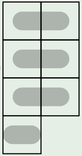
Theorem (Eğecioğlu–Remmel 1990)

$$K^{-1}_{\lambda,\mu} = \sum_{T \in \mathrm{SRT}_{\lambda,\mu}} \mathrm{sgn}(T)$$

Example ($K^{-1}_{(421),(2221)} = 0$)

$(-1)^2$  $+(-1)^3$ 

Example ($K^{-1}_{(2221),(2221)} = 1$)

$(-1)^0$ 

$\mathrm{sgn}(T) = (-1)^{\#\text{ of rows crossed}}$

Matrix problems

Eğecioğlu–Remmel: Can we prove these identities combinatorially?

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{K} \cdot \underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{K^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{K^{-1}} \cdot \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{K} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{I}$$

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Eğecioğlu–Remmel provided a combinatorial proof of $KK^{-1} = I$, but left $K^{-1}K = I$ as an open problem.

Proving $K^{-1}K = I$ combinatorially

$$31 \quad \underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{K^{-1} \leftrightarrow \mathrm{SRT}_{\lambda,\rho}} \overset{\quad\quad\quad 211}{\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{K \leftrightarrow \mathrm{SSYT}_{\rho,\mu}}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \textcircled{0} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{I}$$

$$0 = \sum_{\rho \vdash 4} K^{-1}_{31,\rho} K_{\rho,211} = \sum_{(T,S)} \mathrm{sgn}(T)$$

where the sum is over all pairs (T, S) of SRT T of content $\lambda = 31$ and SSYT S of content $\mu = 211$ of the same shape.

Proving $K^{-1}K = I$ combinatorially

Problem

Let $\lambda \neq \mu$. To show

$$0 = \sum_{\rho \vdash n} K^{-1}_{\lambda,\rho} K_{\rho,\mu} = \sum_{(T,S)} \mathrm{sgn}(T)$$

combinatorially, construct a sign-reversing involution on the set of pairs (T, S) where

- T is a special rim hook tableau of content λ
- S is a semistandard Young tableau of content μ
- T and S have the same shape.

(ie an involution s.t. if $(T, S) \mapsto (V, U)$ then $\mathrm{sgn}(V) = -\mathrm{sgn}(T)$)

For $\lambda = \mu$, it is enough to note that there is only one pair (T, S).

Sign-reversing involution of Sagan and Lee (2006)

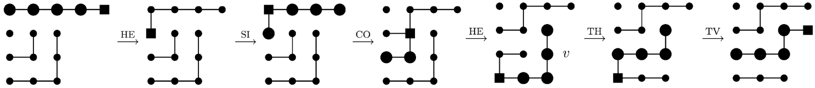
An algorithmic sign-reversing involution
for special rim-hook tableaux

Bruce E. Sagan [a,*], Jaejin Lee [b]

Comb. proof of $K^{-1}K$ for final column (Sagan–Lee (2006))

SL construct a sign-reversing involution on set of pairs (T, S) where S has content (1^n).

SL involution: construct a sequence of *overlapping rooted special rim-hook tableaux* by applying certain involutions at each step, ending with a special rim-hook tableaux of opposite sign.



Application: solution to special case of Stanley–Stembridge e-positivity conjecture for chromatic symmetric functions

Noncommutative symmetric functions (GKLRT 1995)

Recall: $\mathbf{Sym} = \mathbb{C}[h_1, h_2, \ldots]$. The h_i are *commuting variables*.

Noncommutative symmetric functions (NSym)

$\mathbf{NSym} = \mathbb{C}\langle H_1, H_2, \ldots \rangle$, where $H_1, H_2, \ldots,$ is a set of *noncommuting variables* with $\deg(H_i) = i$ for each i.

Complete homogeneous noncommutative symmetric functions

$H_1, H_2, \ldots,$ with $H_0 = 1$.

$\mathbf{NSym}_n = \{\mathbf{NSym}$ functions of homogeneous degree $n\}$

Basis for $\mathbf{NSym}_n$

For $\alpha = (\alpha_1, \ldots, \alpha_\ell) \vDash n$ (α is a composition of n),
$H_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_\ell}$

forgetful map: $\mathbf{NSym} \to \mathbf{Sym}$, $H_\alpha \mapsto h_\alpha$.

Jacobi–Trudi identities

Recall the Jacobi–Trudi identity : $s_\lambda = \det(h_{\lambda_i - i + j})$

$$s_{211} = \begin{vmatrix} h_2 & h_3 & h_4 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{vmatrix} = h_4 - h_{31} - h_{22} + h_{211} = \sum_\lambda K^{-1}_{\lambda,211} h_\lambda$$

(BBSSZ 2014) Immaculate Function $\mathfrak{S}_\alpha := \mathfrak{det}(H_{\alpha_i - i + j})$

$$\mathfrak{S}_{121} = \begin{vmatrix} H_1 & H_2 & H_3 \\ H_1 & H_2 & H_3 \\ 0 & 1 & H_1 \end{vmatrix} = H_{121} - H_{13} - H_{211} + H_{31}$$

Note: $\chi(\mathfrak{S}_\alpha) = s_\alpha$

Immaculate Tableaux (BBSSZ 2014)

Immaculate tableaux (IT) of shape α: a filling of the diagram of α by positive integers that has weakly increasing rows and strictly increasing **1st column**.

1	2	10		
2	8	9	9	21
6	7			
21				

Note: A semistandard Young tableau is an immaculate tableau.

Immaculate Tableaux (BBSSZ 2014)

Immaculate tableaux (IT) of shape α: a filling of the diagram of α by positive integers that has weakly increasing rows and strictly increasing **1st column**.

1	2	10		
2	8	9	9	21
6	7			
21				

Note: A semistandard Young tableau is an immaculate tableau.

Noncommutative Kostka number $\tilde{K}_{\alpha,\beta}$: the number of immaculate tableaux of shape α and content β.

Noncommutative Kostka matrix $\tilde{K} = (\tilde{K}_{\alpha,\beta})$.

Theorem (BBSSZ 2014)

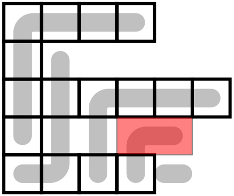
$$H_\beta = \sum_{\beta \models n} \tilde{K}_{\alpha,\beta} \mathfrak{S}_\alpha \qquad \mathfrak{S}_\beta = \sum_{\alpha \models n} \tilde{K}^{-1}_{\alpha,\beta} H_\alpha$$

NSym Kostka Matrix

$$\tilde{K} = \begin{array}{c|cccc} & 3 & 21 & 12 & 111 \\ \hline 3 & 1 & 1 & 1 & 1 \\ 21 & 0 & 1 & 1 & 2 \\ 12 & 0 & 0 & 1 & 1 \\ 111 & 0 & 0 & 0 & 1 \end{array} \qquad \tilde{K}^{-1} = \begin{array}{c|cccc} & 3 & 21 & 12 & 111 \\ \hline 3 & 1 & -1 & 0 & 1 \\ 21 & 0 & 1 & -1 & -1 \\ 12 & 0 & 0 & 1 & -1 \\ 111 & 0 & 0 & 0 & 1 \end{array}$$

What is a combinatorial interpretation of $\tilde{K}^{-1}$?

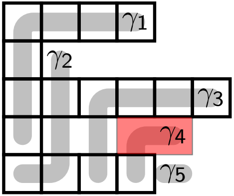
Tunnel Hook Coverings (Allen–Mason 2023)



Tunnel hook covering: a covering of a composition diagram by lattice paths (tunnel hooks) going down and left such that

1. there is a tunnel hook starting in each row (use next available cell if needed),
2. tunnel hooks may exit the diagram,
3. every time a tunnel hook covers a cell not in the diagram nor its starting row, it makes a negative cell later in that row, and
4. all negative cells are covered by tunnel hooks starting in that row

Tunnel Hook Coverings (Allen–Mason 2023)



Let γ_i be the tunnel hook starting in row i

- content $\alpha = (\alpha_1, \alpha_2, \dots)$:
  $\alpha_i = \#\{\text{cells covered by } \gamma_i\} - \#\{\text{nondiagram cells covered in row } i\}$
  $\alpha = (7, 4, 6, -1, 0)$
- shape: β shape of the diagram.
  $\beta = (4, 1, 6, 1, 4)$
- sign: $\text{sgn}(T) = (-1)^k = (-1)^9 = -1$,
  $k = \#\{\text{rows crossed}\} = 9$

Tunnel Hook Coverings (Allen–Mason 2023)

$\mathsf{THC}_{\alpha,\beta} = \{$tunnel hook coverings of content α and shape $\beta\}$

Theorem (Allen–Mason 2023)

For all compositions α, β, we have

$$\tilde{K}^{-1}_{\alpha,\beta} = \sum_{T \in \mathsf{THC}_{\alpha,\beta}} \mathrm{sgn}(T).$$

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Lift of matrix problems to NSym

Prove combinatorially that $\tilde{K}\tilde{K}^{-1} = I$ and $\tilde{K}^{-1}\tilde{K} = I$.

We construct sign-reversing involutions to combinatorially prove

- $\tilde{K}^{-1}\tilde{K} = I$ (in this talk)
- $\tilde{K}\tilde{K}^{-1} = I$ (in full paper, coming soon!)

Proving $\tilde{K}^{-1}\tilde{K} = I$ combinatorially (Allen–C–Mason '25+)

$$\begin{array}{c} \\ \alpha{=}21 \\ \\ \\ \end{array} \underbrace{\begin{bmatrix} 1 & -1 & 0 & 1 \\ \color{red}{0} & \color{red}{1} & \color{red}{-1} & \color{red}{-1} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\tilde{K}^{-1}} \cdot \underbrace{\overset{\beta=12}{\begin{bmatrix} 1 & 1 & \color{red}{1} & 1 \\ 0 & 1 & \color{red}{1} & 2 \\ 0 & 0 & \color{red}{1} & 1 \\ 0 & 0 & \color{red}{0} & 1 \end{bmatrix}}}_{\tilde{K}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \bigcirc\!\!\!\!\color{red}{0} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$0 = \sum_{\delta} \tilde{K}^{-1}_{\alpha,\delta}\tilde{K}_{\delta,\beta} = \sum_{(T,S)} \mathrm{sgn}(T)$$

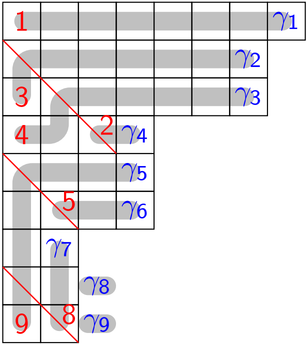
where the sum is over the set of (T, S) where

- T is a tunnel hook covering of content α
- S is an immaculate tableau of content β
- T and S have the same shape.

Problem

Construct a sign-reversing involution on this set of pairs (T, S).

Permutations and Tunnel Hook Coverings (AM 2023)

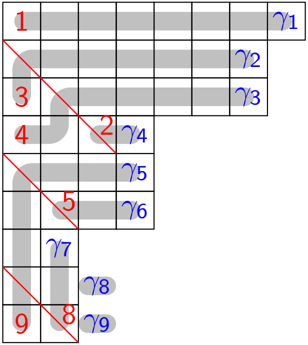


The *j*-th diagonal of a composition diagram are the cells $(1, j), (2, j+1), \ldots$

The permutation $\pi = \pi(T)$ of a tunnel hook covering T is defined by $\pi_i = j$ if γ_i ends on diagonal j.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 4 & 2 & 9 & 5 & 8 & 6 & 7 \end{pmatrix}$$

Permutations and Tunnel Hook Coverings (AM 2023)



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$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 4 & 2 & 9 & 5 & 8 & 6 & 7 \end{pmatrix}$$

If $\beta = (\beta_1, \ldots, \beta_\ell)$, $T \mapsto \pi(T)$ is a bijection $\bigsqcup_\alpha \mathrm{THC}_{\alpha,\beta} \to \mathfrak{S}_\ell$ s.t.

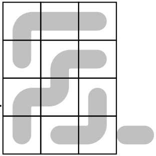
$$\mathrm{sgn}(\pi(T)) = \mathrm{sgn}(T)$$

Idea: sign-reversing involution $\leftrightarrow$ multiplying by transposition.

Proving $\tilde{K}^{-1}\tilde{K} = I$ combinatorially (Allen–C–Mason '25+)

$T =$

$\pi = 2431$



$S =$

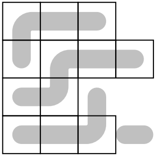
1	1	2
2	5	5
3	4	5
4	4	4

$m = 5$

$r = 3$

1. r: row with $\max(S) = m$ s.t. if m is in row i, then $\pi(i) \geq \pi(r)$
2. If $\pi(r) = r$ and m only appears in row r:
   1. Remove final row of T, S, induct, and reattach
3. Otherwise,
   1. S': move m to row $\pi^{-1}(\pi(r)+1) = 2$.
   2. T': $\pi(T') = (\pi(r), \pi(r)+1)\pi = 2341$

$T' =$



$S' =$

1	1	2	
2	5	5	5
3	4		
4	4	4	

Since $\pi(T') = (\pi(r), \pi(r)+1)\pi$ where $\pi = \pi(T)$, we have $\operatorname{sgn}(T') = -\operatorname{sgn}(T)$.

Theorem (Allen–C–Mason '25+)

The map $\psi : \bigsqcup_\delta \mathrm{THC}_{\alpha,\delta} \times \mathrm{IT}_{\delta,\beta} \to \bigsqcup_\delta \mathrm{THC}_{\alpha,\delta} \times \mathrm{IT}_{\delta,\beta}$ *defined by* $\psi(T, S) = (T', S')$ *is a sign-reversing involution for any* $\alpha \neq \beta$*.*

Proving $\tilde{K}^{-1}\tilde{K} = I$ combinatorially (Allen–C–Mason '25+)

Since $\pi(T') = (\pi(r), \pi(r)+1)\pi$ where $\pi = \pi(T)$, we have $\operatorname{sgn}(T') = -\operatorname{sgn}(T)$.

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The map $\psi : \bigsqcup_\delta \mathrm{THC}_{\alpha,\delta} \times \mathrm{IT}_{\delta,\beta} \to \bigsqcup_\delta \mathrm{THC}_{\alpha,\delta} \times \mathrm{IT}_{\delta,\beta}$ *defined by* $\psi(T, S) = (T', S')$ *is a sign-reversing involution for any* $\alpha \neq \beta$*.*

We will use our involution for $\tilde{K}^{-1}\tilde{K} = I$ to construct an involution for $K^{-1}K = I$.

Reduction to **Sym**–Two important notes

Note 1

Every semistandard Young tableau is an immaculate tableau.

Note 2

Since $\chi(H_\alpha) = h_\alpha$, for partitions $\lambda, \mu \vdash n$, we have

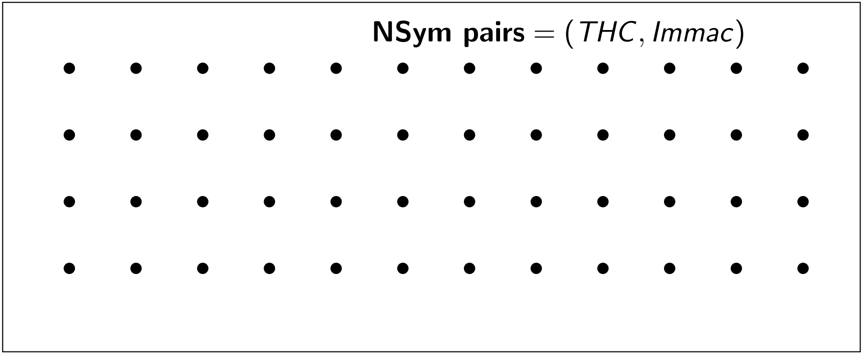
$$K^{-1}_{\lambda,\mu} = \sum_{\substack{\alpha \models n \\ dec(\alpha)=\lambda}} \sum_{T \in \mathrm{THC}_{\alpha,\mu}} \mathrm{sgn}(T)$$

where $dec(\alpha)$ is weakly decreasing rearrangement of λ.

Thus, tunnel hook covering provide a combinatorial interpretation of the (**Sym**) inverse Kotska matrix K^{-1}.

Combinatorial proof that $K^{-1}K = I$ (sketch) (ACM '25+)

For fix λ, μ, consider the collection of all pairs (T, S) where T is THC of content λ, S is immaculate tableau content μ, and S and T have the same shape (**NSym pairs**).



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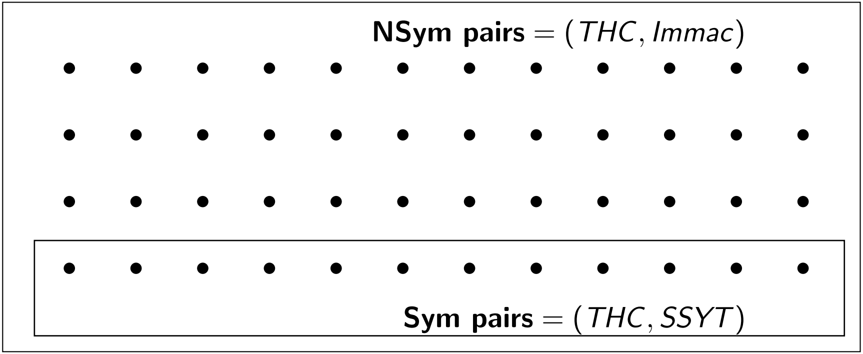
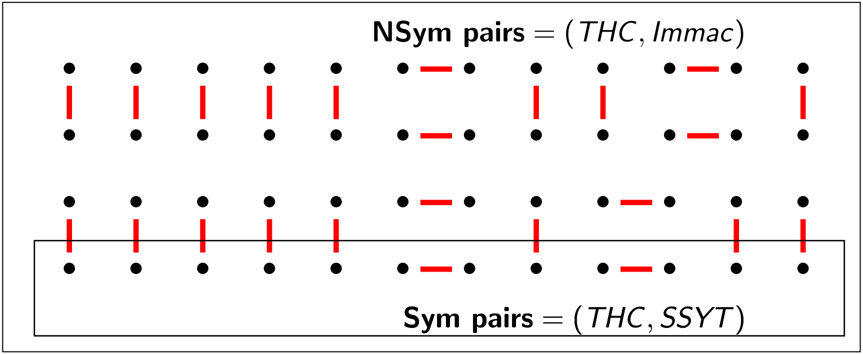
The set of $(THC, SSYT)$ pairs (**Sym pairs**) is contained in the set of **NSym pairs**.



Combinatorial proof that $K^{-1}K = I$ (sketch) (ACM '25+)

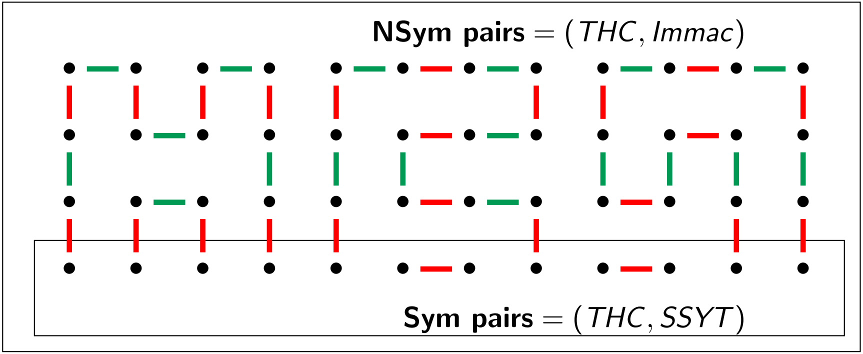
Our involution on **NSym pairs** is in red.

Note that this does not always take **Sym pairs** to **Sym pairs**.



Combinatorial proof that $K^{-1}K = I$ (sketch) (ACM '25+)

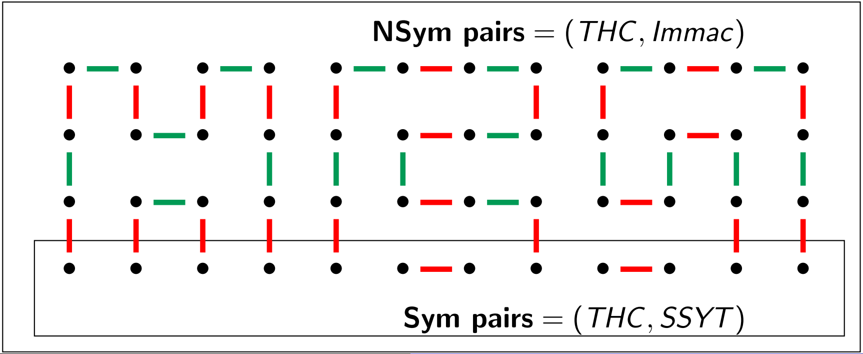
We introduce a new sign-reversing involution in green on the set of **NSym pairs** that are not **Sym pairs** distinct from our involution



Combinatorial proof that $K^{-1}K = I$ (sketch) (ACM '25+)

Thus, the set of **NSym pairs** forms a graph.

- **Sym pairs** have degree 1 and the rest have degree 2.
- Thus, any component with **Sym** pair is a path starting and ending in **Sym**. This defines an involution on the **Sym pairs**.
- Since only the red involution applies to **Sym pairs**, these path have odd length, so it is sign-reversing. □



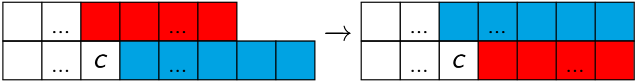
Green involution

Let $E_{\lambda,\mu} =$ **NSym pairs** $\setminus$ **Sym pairs**
For $(T, S) \in E_{\lambda,\mu}$, say cell c of S is **bad** if either

- The cell above c is empty and c is not in the first row
- The cell above c contains a weakly larger element than in c

Since (T, S) has S immaculate, but not SSYT, it has a bad cell.

Let j be the leftmost column of S containing a bad cell and let i be the largest value such that row i contains a bad cell in column j. Swap the following cells to create S' where $c = (i, j)$.



Define T' to be THC with permutation
$\pi(T') = \pi(T)(\pi(i) - 1, \pi(i))$

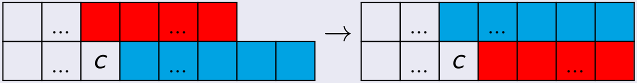
Green involution - Inspiration

Green=G for Gessel–Viennot or Gasharov

A cell c of S is **bad** if either

- The cell above c is empty and c is not in the first row
- The cell above c contains a "larger" element than in c

Let j be the leftmost column of S containing a bad cell and let i be the largest value such that row i contains a bad cell in column j. Swap the following cells to create S' where $c = (i, j)$.

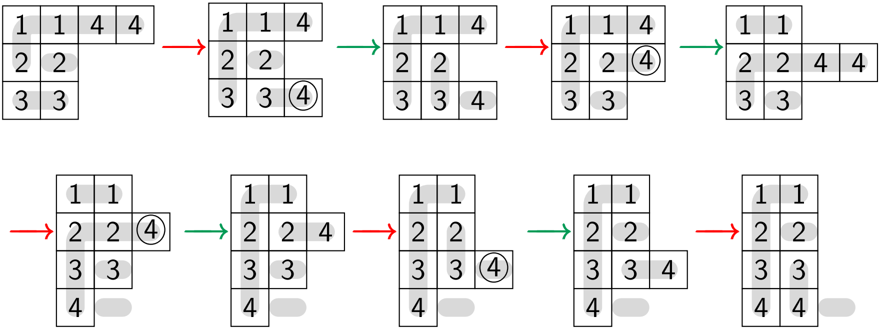


This involution is spiritually the same involution that appears in work of **Gessel**–Viennot on combinatorial determinants (1989) and **Gasharov** on chromatic symmetric functions (1996), among other places.

Sign-reversing involution for $K^{-1}K = I$

Theorem (Allen–C.–Mason (2025+))

The described map on the **Sym pairs** *is a sign–reversing involution.*



Future Directions

1. Apply THC techniques to study $e/s/\mathfrak{S}$-positivity of important families of symmetric functions e.g. chromatic quasisymmetric functions, Macdonald polynomials
2. Look for other involutions that fit into this framework.
3. Product formulas for dual immacualates (and others)

Thanks!