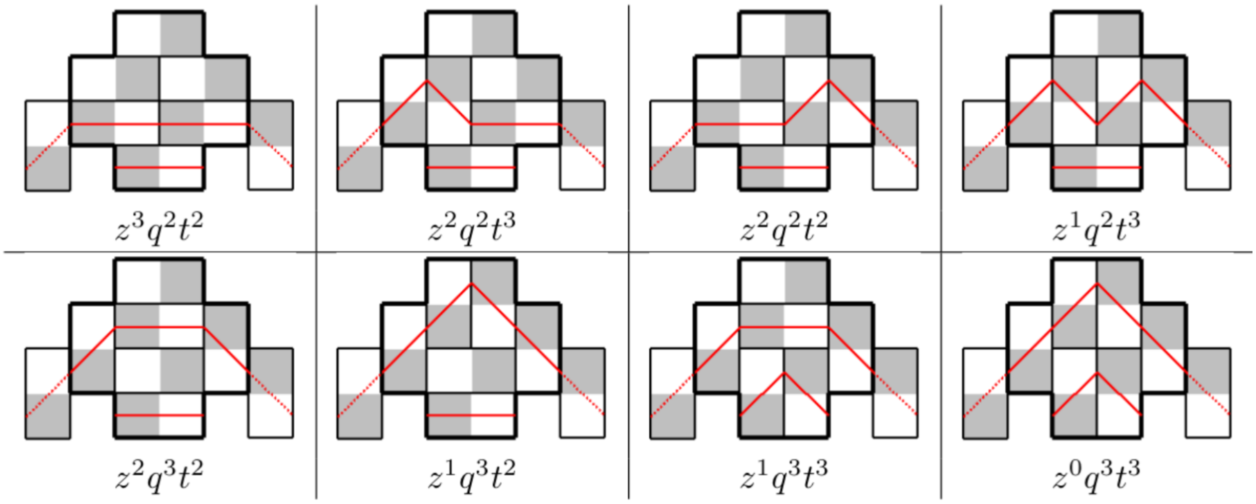
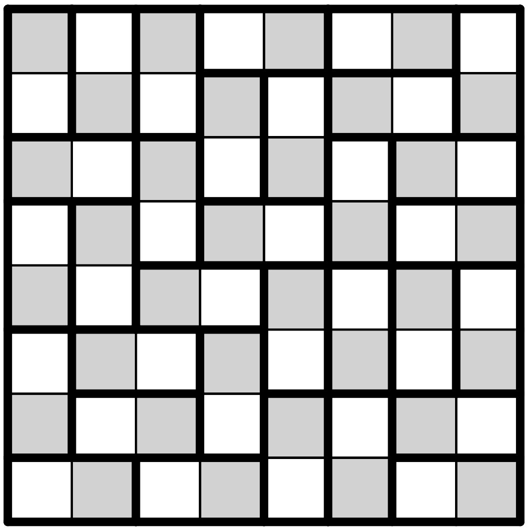
Aztec Diamond Domino Tilings and Macdonald Theory

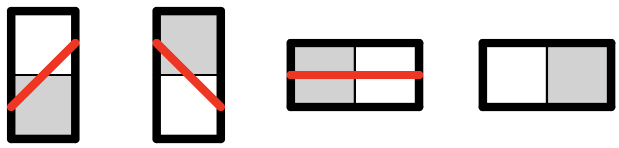
Joint with Yi-Lin Lee

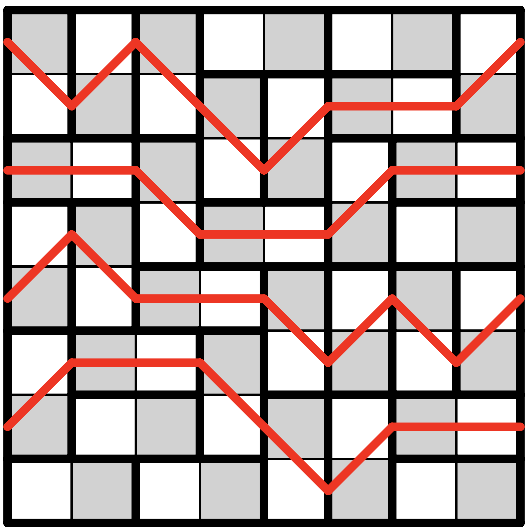


Given a domino tiling of a region,



- checkerboard color the region.
- decorate each of the four types of dominos with a path step:





This produces a family of non-intersecting paths in the region.

From left-to-right, the paths enter at shaded cells and exit at unshaded cells.

This gives a bijection:

Domino tilings $\leftrightsquigarrow$ Non-intersecting families of paths

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Idea: Use tools from domino tilings to study "q,t-Catalan"-type objects, coming from the study of Macdonald polynomials.

The <u>q,t-Catalan numbers</u> are polynomials $C_n(q,t) \in \mathbb{N}[q,t]$ that refine the ordinary Catalan numbers:

$$C_n(1,1) = C_n := \frac{1}{n+1}\binom{2n}{n}.$$

n	1	2	3	4	5	$\cdots$
C_n	1	2	5	14	42	$\cdots$

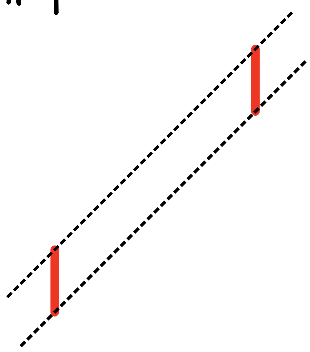
$C_1(q,t) = 1 \qquad C_2(q,t) = q + t$

$$C_3(q,t) = q^3 + q^2 t + qt + qt^2 + t^3 \quad \leftrightarrow \quad \begin{array}{c|l} q \uparrow & \begin{matrix} 1 & & & \\ 0 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 0 & 0 & 1 \end{matrix} \\ \hline & \quad \xrightarrow{\quad t \quad} \end{array}$$

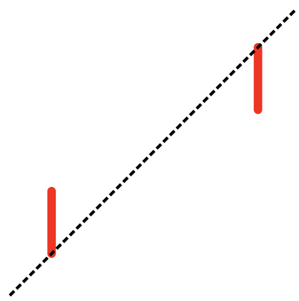
$$C_n(q,t) = \sum_{\substack{\text{Dyck paths} \\ \pi \text{ of order } n}} q^{\operatorname{area}(\pi)} t^{\operatorname{dinv}(\pi)}$$

where $\operatorname{area}(\pi)$ = # boxes below π and above the diagonal

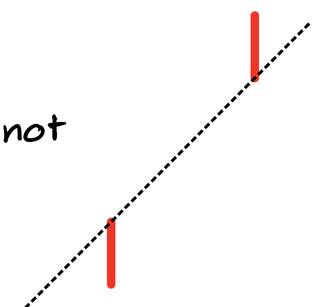
$\operatorname{dinv}(\pi)$ = # pairs of steps in π arranged as



or



(but not



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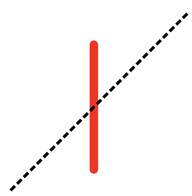
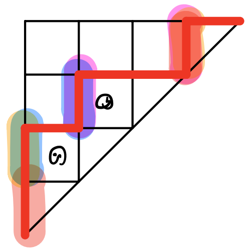
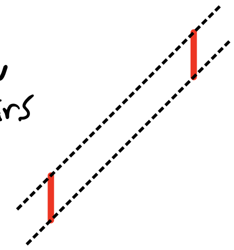
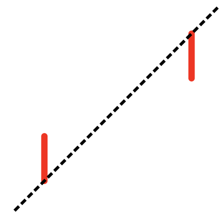
)

Example

$\pi =$  is one of the 14 Dyck paths of order 4.

$\text{area}(\pi) = 2$

$\text{dinv}(\pi) = 4$

(For dinv count pairs like  or )

So contributes a term

$$C_4(q,t) = \cdots + q^2 t^4 + \cdots$$

Symmetric Functions ⇝ generalizations of $C_n(q,t)$.

$\Lambda =$ ring of symmetric functions in $x_1, x_2, \ldots$ with coefficients in $\mathbb{Q}(q,t)$.

Λ has many (vector space) bases:

- m_λ (monomial)
- e_λ (elementary)
- h_λ (complete homogeneous)
- s_λ (Schur)
- $\tilde{H}_\lambda$ (Modified Macdonald)

  ⋮

It is an ongoing area of research to understand the "change of basis matrices" between $\tilde{H}_\lambda$ and the more common bases.

The "nabla operator" is a roundabout way to encode this information:

$\nabla : \Lambda \to \Lambda$ is the q,t-linear map extending

$$\tilde{H}_\lambda \mapsto q^{n(\lambda')} t^{n(\lambda)} \tilde{H}_\lambda$$

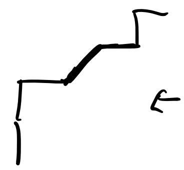
where λ' is the conjugate partition to λ

and $n(\mu) = \sum_i (i-1)\mu_i$.

<u>Theorem</u> (Garsia, Haglund, '00)

$$\langle \nabla(e_n), e_n \rangle = C_n(q,t)$$

<u>Cor</u> $C_n(q,t) = C_n(t,q)$ (!)



← Schröder

$\langle \nabla(e_n), e_n \rangle$ = q,t-Catalan ⇜ $\langle \nabla(s_\lambda), e_n \rangle = (\pm 1)\cdot q,t$ - nested Dyck paths

↑ ↑

$\langle \nabla(e_n), h_d e_{n-d} \rangle$ = q,t-Schröder with d-diagonals ⇜ $\boxed{\langle \nabla(s_\lambda), h_d e_{n-d} \rangle = q,t\text{-nested Schröder paths}}$

↑ ↑

Full monomial expansion of $\nabla(e_n)$ = q,t-labelled Dyck paths ⇜ Full monomial expansion of $\nabla(s_\lambda)$ = q,t-labelled nested Dyck paths

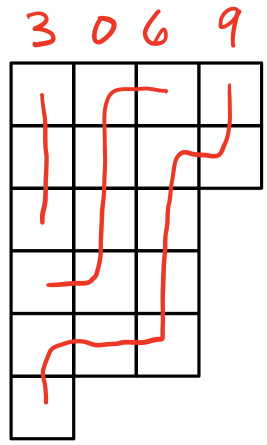
"The shuffle formula" (CM '18)

"Loehr-Warrington formula" (BHMPS '21)

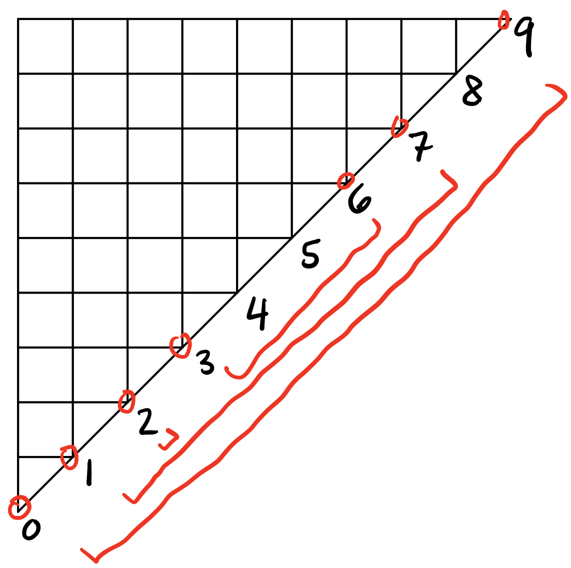
A <u>λ-family of paths</u> is a set of non-intersecting paths with starting points $(0,0),(1,1),(2,2),\ldots$ and ending points determined by the border strip decomposition of λ

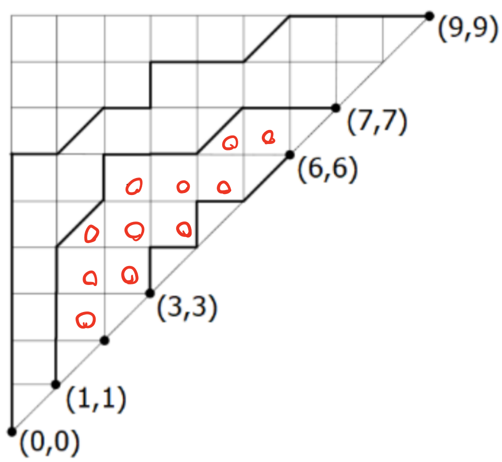
<u>example</u>

$\lambda = (4,4,3,3,3,1) \leftrightsquigarrow$



a λ family:





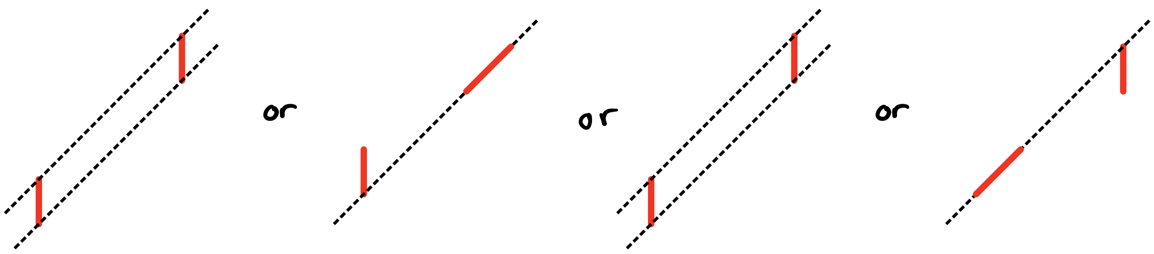
The <u>area</u> of a Schröder path is the number of triangles $\triangle$ below the path and above the diagonal.

The <u>area</u> of a family of paths is the sum of the areas of the individual paths.

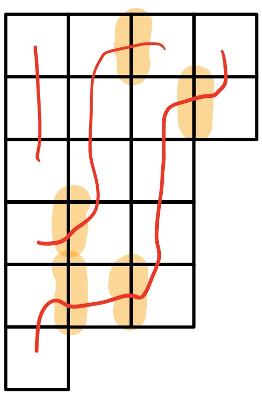
e.g. above has area $27 + 11 + 0 + 0 = 38$

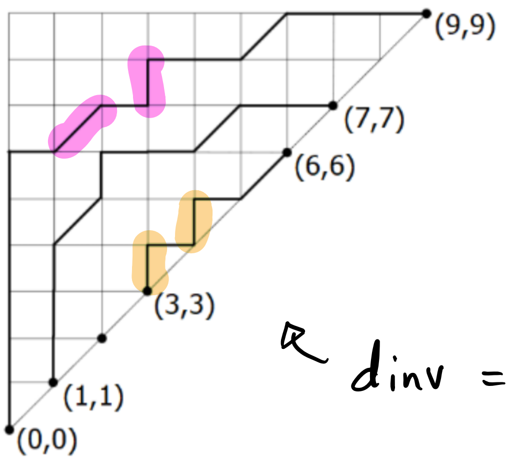
The <u>dinv</u> of a (λ-family of) Schröder paths is

adj(λ) + # pairs of steps



adj(λ) = # times a border strip crosses a vertical wall





dinv = 5 + 29

adj(λ) = 5

Let
$$P_\lambda(z;q,t) = \sum_{\substack{\pi \text{ a } \lambda\text{-family} \\ \text{of Schröder} \\ \text{paths}}} z^{\#\text{diagonals}(\pi)} q^{\text{area}(\pi)} t^{\text{dinv}(\pi)}$$

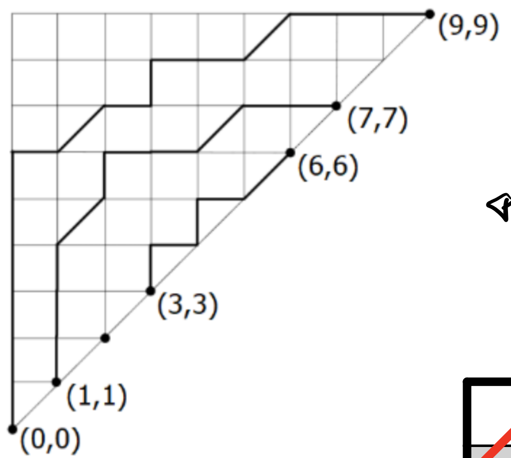
<u>Theorem</u> (C., Lee '25).

For any partition $\lambda \vdash n$,
$$P_\lambda(z;q,t) = (-1)^{\text{adj}(\lambda)} \sum_{d=0}^{n} z^d \langle \nabla(s_\lambda), h_d e_{n-d} \rangle$$

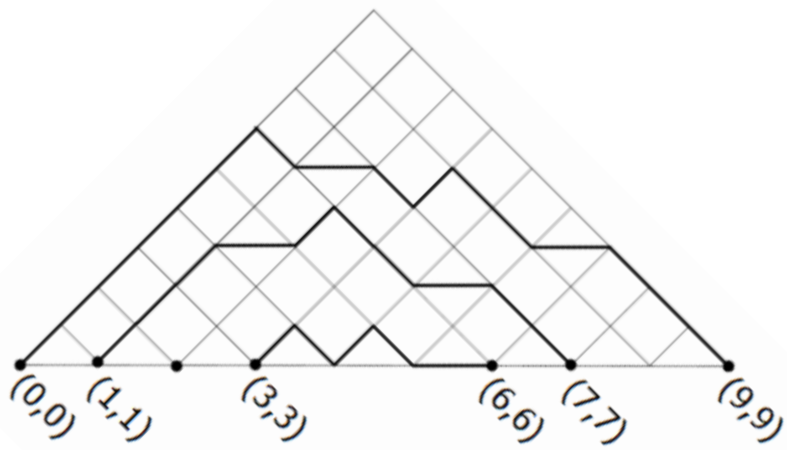
ie. these statistics on nested Schröder paths are the "correct analogues" of those from q,t-Catalan numbers

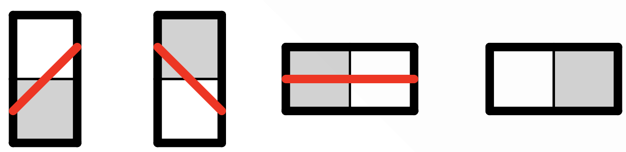
$\underline{\text{Cor}}$ $P_\lambda(z;q,t) = P_\lambda(z;t,q)$ (!)

Is there a region R_λ whose domino tilings $\leftrightarrow$ λ-families of Schröder paths?

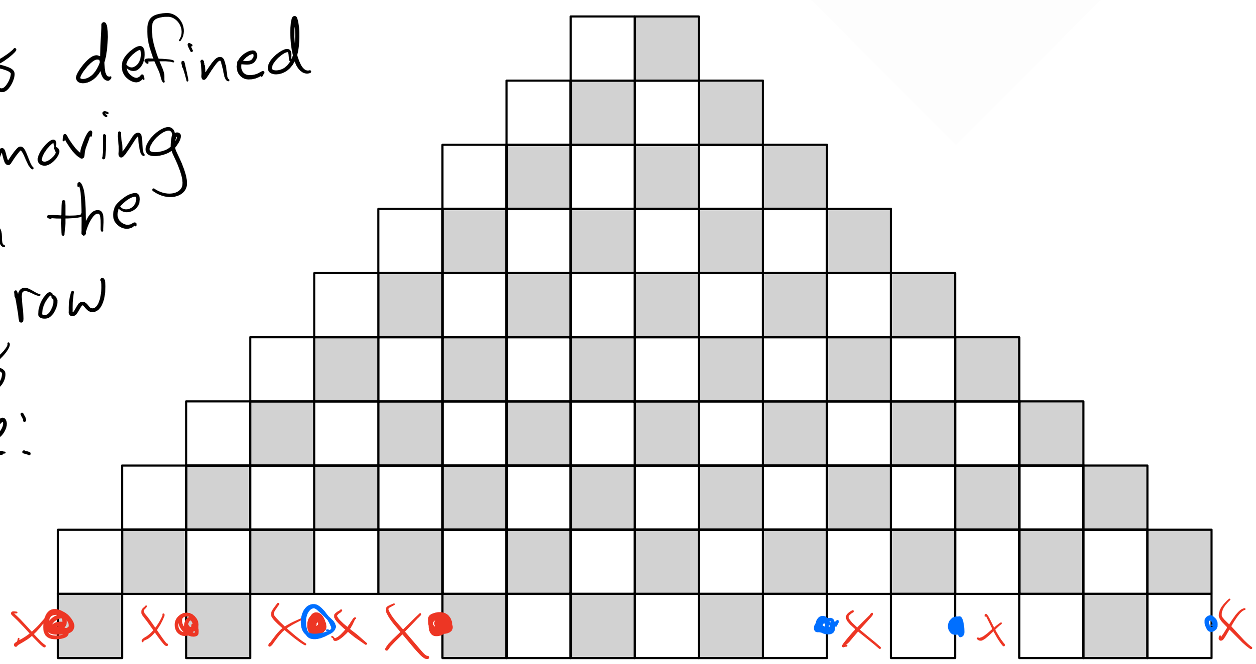


$\leftrightarrow$

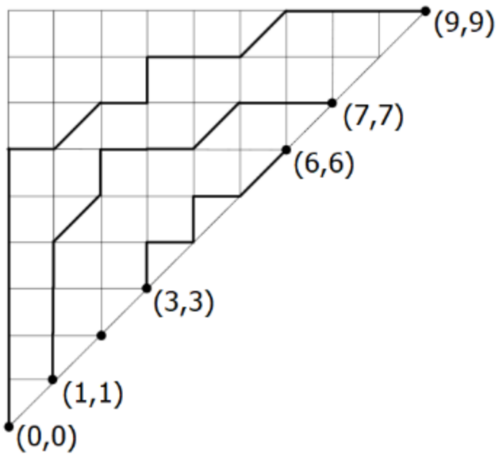




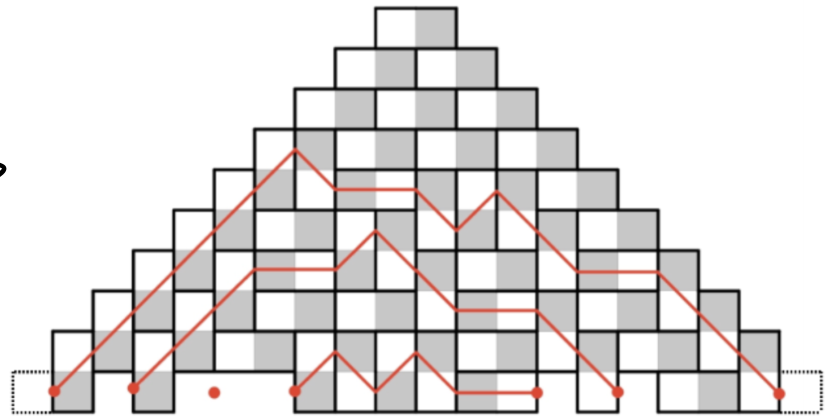
R_λ is defined by removing cells in the bottom row of this triangle:



e.g.

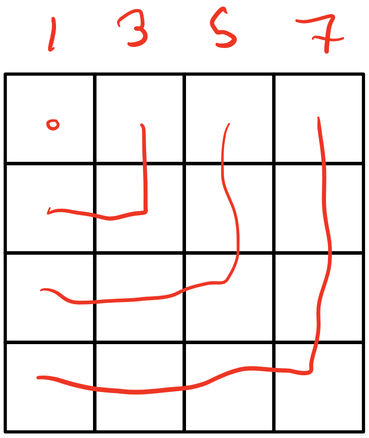


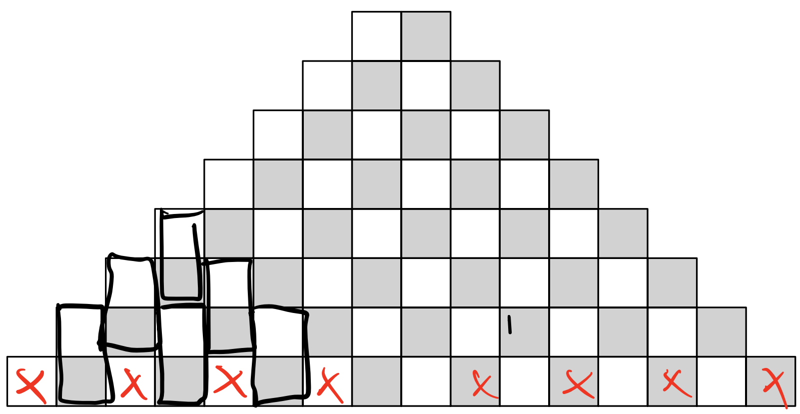
$\longleftrightarrow$



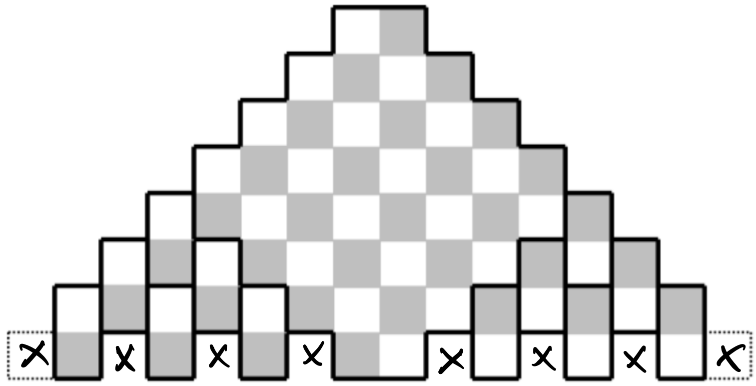
Do we learn anything new about P_λ by thinking about it as a sum over domino tilings instead of paths?

Consider $\lambda = (n^n)$ a square shape. $\lambda = (4^4)$





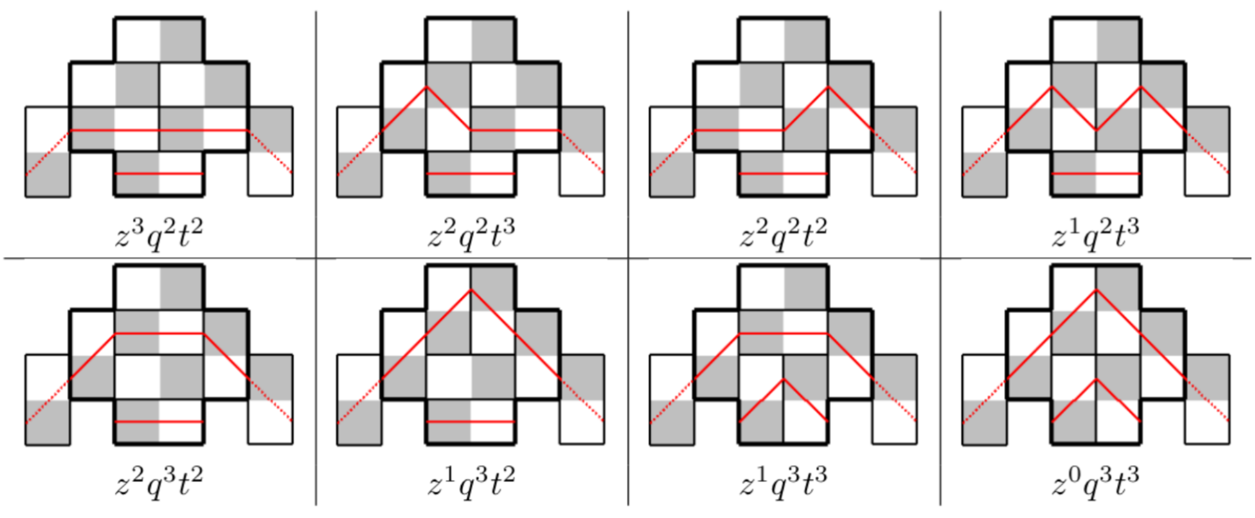
λ-families of Schröder paths $\longleftrightarrow$ Domino tilings of the <u>Aztec</u> Diamond



Set $AD_n(z;q,t) = P_{(n^n)}(z;q,t)$

The number of Domino tilings of AD_n is $2^{\binom{n+1}{2}}$, can we see this in $AD_n(z;q,t)$?

e.g. $n=2$, there are $2^3 = 8$ tilings:



$$AD_2(z;q,t) = q^2t^2(z+1)(z+q)(z+t)$$

Theorem (C., Lee '25)

$$AD_n(z;q,t) = (qt)^{n^2(n-1)/2} \prod_{\substack{a,b\geq 0 \\ a+b<n}} (z+q^a t^b)$$

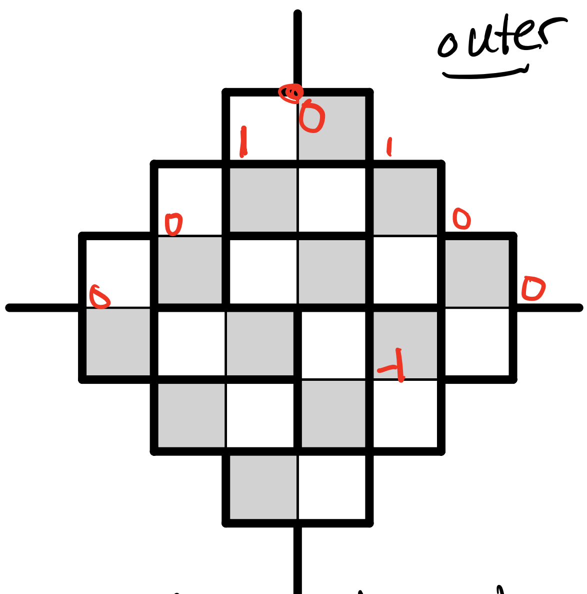
In particular, the $q \leftrightarrow t$ symmetry is clear!

We prove this formula by induction using domino shuffling.

We don't know how to prove it from the symmetric function side!

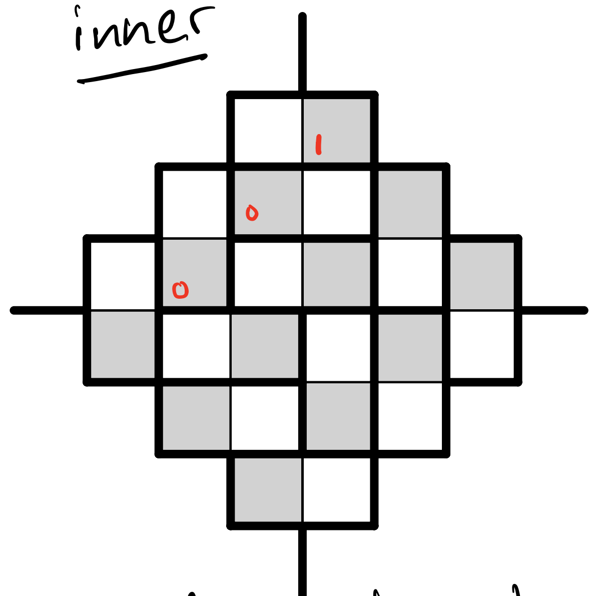
Every domino tiling of AD_n has two associated matrices:

outer



order	label
2	4
3	0
4	1

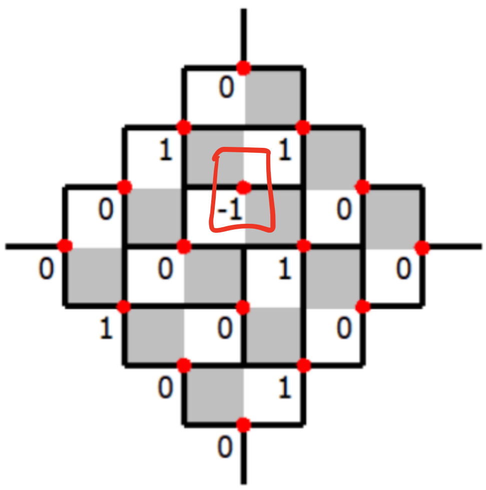
inner

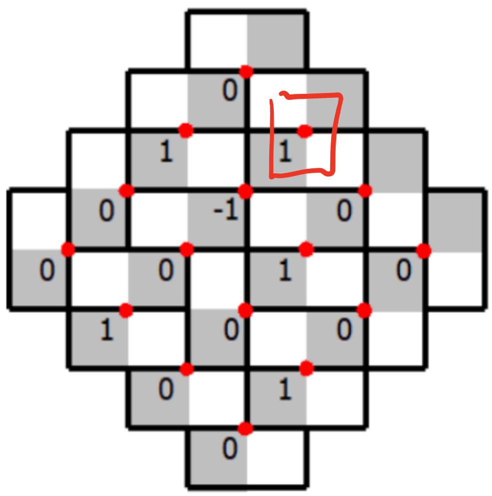


order	label
2	1
3	0
4	-1

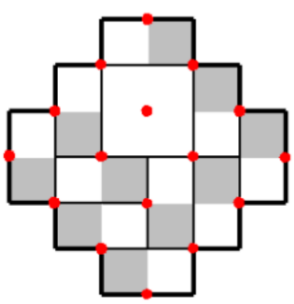
It turns out both associated matrices are <u>Alternating Sign Matrices</u>.

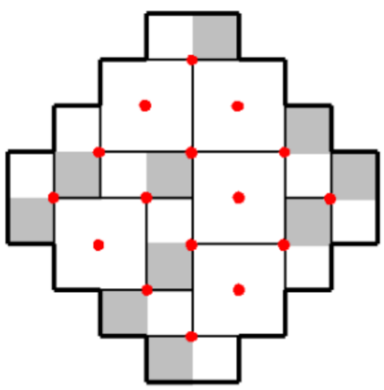
Domino shuffling $\longleftrightarrow$ Fixing an ASM and grouping together tilings of AD_n and AD_{n+1} that have that matrix as outer/inner ASM

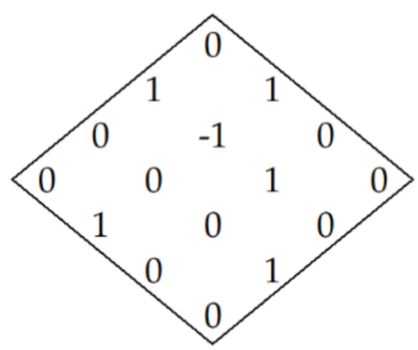




The ASM determines the tilings up to orientations of 2×2 blocks ~~and~~ -1's and 1's.







The weighted sum over tilings with <u>outer</u> ASM A

is of the form $\displaystyle q^* t^* z^* \prod_{-1 \in A} (qz + t^*)$

The weighted sum over tilings with inner ASM A

is of the form $\displaystyle q^* t^* z^* \prod_{+1 \in A} (qz + t^*)$

where the exponents are some specific statistics on A.

The ratio of these is independent of A(!)

$\rightsquigarrow$ Formula follows by induction.

For non-square partitions, the formulas are not so simple.

We <u>conjecture</u> that

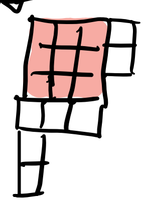
$$AD_k(z;q,t) \text{ divides } P_\lambda(z;q,t)$$

and the quotient has nonnegative integer coefficients, where

k = # nonzero-length paths in a λ-family

= size of the largest square in λ:

e.g. $k=3$



Thank You!