# NEARLY TORIC VARIETIES



(Mahir Bilen Can, Nestor Diaz Morera)

Let $X$ be an algebraic variety which admits an algebraic action of the group,

$$L := GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C}) \times \cdots \times GL(n_k, \mathbb{C}), \quad \sum n_i = n$$

We will assume that $X$ is not terribly singular. For example, assuming that $X$ is normal is good. We want to keep track of the restriction of this action to two special subgroups.

① $B_L := \underbrace{B(n_1, \mathbb{C})} \times B(n_2, \mathbb{C}) \times \cdots \times B(n_k, \mathbb{C})$.

$\qquad\quad \cup| \qquad\qquad\quad \cup| \qquad\qquad\quad \cup|$

② $T_L := T(n_1, \mathbb{C}) \times T(n_2, \mathbb{C}) \times \cdots \times T(n_k, \mathbb{C})$.

Here, $B(m, \mathbb{C})$ is the group of all upper triangular matrices from $GL(m, \mathbb{C})$, $T(m, \mathbb{C})$ is the group of all diagonal matrices from $GL(m, \mathbb{C})$.

[Note. $T_L = T(n_1 + \cdots + n_k, \mathbb{C})$] $T_L :=$ maximal torus in $GL(n, \mathbb{C})$

Definition: If there is a $\overbrace{\text{0-codimensional}}^{\text{dim. of orbit} = \dim X}$ $B_L$-orbit in $X$, then we call $X$ a spherical $L$-variety.

If there is a 0-codimensional $T_L$-orbit in $X$, then we call $X$ a toric variety.

- The theory of spherical varieties is well-developed.
- The theory of toric varieties can be viewed as a special case of the theory of spherical varieties.

  (Choose $n_1 = n_2 = \cdots = n_k = 1$.)
- There are important representation theoretic reasons for caring about spherical varieties.

  (The coordinate ring of a spherical $L$-variety is a multiplicity free representation of $L$.)

More generally, we can speak about the complexities of the actions of $B_L$ and $T_L$.

Let $H \in \{B_L, T_L\}$.

$GL(2,\mathbb{C}) \times GL(2,\mathbb{C})$ acts on $\mathrm{Mat}(2,\mathbb{C})$ via $(A,B)\cdot X = AXB^{-1}$

complexity

$B(2,\mathbb{C})w_0B(2,\mathbb{C}) \sqcup B(2,\mathbb{C})\,\mathrm{id}\,B(2,\mathbb{C}) \sqcup \cdots$

The <u>generic modality</u> of the action of $H$ is the minimum codimension of an $H$-orbit in $X$. $B(2,\mathbb{C})rB(2,\mathbb{C})$

Rosenlicht (1956): The generic modality of the action $H \times X \longrightarrow X$ is equal to the transcendence degree of the field of -invariant rational functions on $X$ over $\mathbb{C}$.

<u>Notation</u>:

$c_H(X)$ = the generic modality of the $H$-action on $X$ (also called the complexity)

minimum codimension of an $H$-orbit in $X$

**Definition:**

Let $X$ be a spherical $L$-variety. If additionally

$$c_{T_L}(X) = 1$$

holds, then we call $X$ a nearly toric variety.

Set-up: The variety of full flags in $\mathbb{C}^n$:

$$Fl(n,\mathbb{C}) = \{ 0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim_{\mathbb{C}} V_i = i \}.$$

$$\cong GL(n,\mathbb{C}) / B(n,\mathbb{C}).$$

A Schubert variety in $Fl(n,\mathbb{C})$ is the closure of a $B(n,\mathbb{C})$-orbit in $GL(n,\mathbb{C})/B(n,\mathbb{C})$.

The Schubert varieties in $Fl(n,\mathbb{C})$ are parametrized by the elements of the symmetric group $S_n$.

This is a consequence of the Bruhat decomposition

$$GL(n,\mathbb{C}) = \bigsqcup_{w \in S_n} B(n,\mathbb{C}) \cdot w \cdot B(n,\mathbb{C}).$$

We will use $X_w$ to denote the Schubert variety defined by

$$\overline{B(n,\mathbb{C}) \cdot w\, B(n,\mathbb{C}) / B(n,\mathbb{C})} \subseteq Fl(n,\mathbb{C}).$$

<u>Fact:</u> Schubert varieties are normal.

# Questions:

1. Which Schubert varieties are spherical?
2. Which Schubert varieties are toric?
3. Which Schubert varieties are nearly toric?

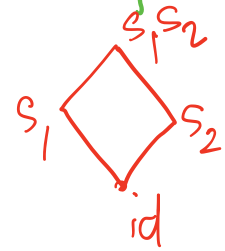
## Lets start with 2.

**Theorem:** (Karuppuchamy, 2011)
Let $w \in S_n$. The Schubert variety $X_w$ is a toric variety if and only if the permutation $w$ is a Coxeter element.

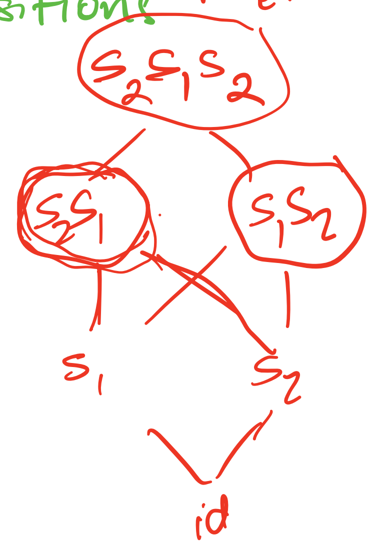
$S = \{s_1, s_2, \dots, s_{n-1}\}$ : simple transpositions

$\Rightarrow w = s_{i_1} \cdots s_{i_{n-1}}$

where $\{i_1, \dots, i_{n-1}\} = \{1, \dots, n-1\}$.




not a Coxeter elt.



We continue with question #1 

Two conjectures of Gao-Hodges-Yong.

(1) $X_w$ is a spherical L-variety iff $w$ can be written in the form $w = w_{0,J} \cdot c$, where $w_{0,J}$ is the longest element of the subgroup

$$\langle s_i \mid i \in J \rangle \subseteq S_n,$$

and $c$ is a Coxeter type element s.t.

$$\ell(w) = \ell(w_{0,J}) + \ell(c).$$

(2) $X_w$ is a spherical L-variety iff $w$ avoids

$$\mathscr{P} := \begin{Bmatrix} 24531 & 25314 & 25341 & 34512 & 34521 & 35412 & 35421 \\ 42531 & 45123 & 45213 & 45231 & 45312 & 52314 & 52341 \\ 53124 & 53142 & 53412 & 53421 & 54123 & 54213 & 54231 \end{Bmatrix}.$$

> the first conjecture was proved independently by Can-Saha and Gao Hodges Yong ('23).

> The second conjecture was proved by Gaetz ('22)

We will now focus on question #3.

Relevant to this discussion are the works of Daniel Daly and Bridget Tenner.

$$\mathcal{A}_n := \{ w \in S_n \mid w \text{ avoids } 3412, \text{ contains } 321 \text{ only once} \}$$

$$\mathcal{B}_n := \{ w \in S_n \mid w \text{ avoids } 321, \text{ contains } 3412 \text{ only once} \}$$

Daly (using Tenner's work) showed that:

(D1) $v \in \mathcal{A}_n$ iff $\exists$ reduced decomposition of $v$ having $s_i\, s_{i+1}\, s_i$ as a factor with no repetetions of simple transpositions.

(D2) $v \in \mathcal{B}_n$ iff $\exists$ reduced decomposition of $v$ having $s_i\, s_{i+1}\, s_{i-1}\, s_i$ as a factor with no repetetions of simple transpositions.

(D3) there is a bijection between $\mathcal{A}_n$ and $\mathcal{B}_{n+1}$.

(Daly's work was around 2010, ..., 2013)

# In $Fl(n, \mathbb{C})$

The importance of these results were recognized by Lee – Park – Masuda who observed that (2021)

(LPM 1) $c_{T_L}(X_w) = 1$ and $X_w$ is smooth iff $w \in \mathcal{A}_n$,

(LPM 2) $c_{T_L}(X_w) = 1$ and $X_w$ is singular iff $w \in \mathcal{B}_n$.

We define:

$$\mathcal{M}_n := \left\{ w \in S_n \;\middle|\; \begin{array}{c} w \text{ avoids } 3412, \text{ contains } 321 \text{ once}, \\ \text{and contains } \boxed{25314} \end{array} \right\}.$$

Theorem: (C.-Diaz 23)

There is a bijection $\Psi : \mathcal{M}_{n+1} \to \mathcal{B}_n$.

(This map analyzes whether $s_n$ appears in $w$ or not.)

Theorem: (C.-Diaz 23)

Let $X_w \subseteq Fl(n,\mathbb{C})$ be s.t. $\underbrace{c_{T_L}(X_w) = 1}$.

Then the following assertions hold:

① $X_w$ is a singular nearly toric variety iff $w \in \mathcal{B}_n$.

② $X_w$ is a smooth nearly toric variety iff $w \in \mathcal{M}_n$.

Theorem: (C.-Diaz 23)

Let $b_n := |\mathcal{B}_n|$, $r_n := |\mathcal{U}_n|$.

Let $F_m$ denote the $m$-th Fibonacci number.

Then we have

Daly

① $$b_n = \frac{2(2n-7)F_{2n-8} + (7n-23)F_{2n-7}}{5},$$

② $$\boxed{r_n = (n-2)F_{2n-4}}.$$

(See Develin–Martin–Reiner paper)



Definition: A Schubert variety $X_w$ is called a partition Schubert variety if $w$ is 312-avoiding.

These Schubert varieties are closely related to the rook placements in Young diagrams.

Question: What can we say about the toric, nearly-toric, or sphericality properties of such varities?

A useful combinatorial method here is due to Bandlow – Killpatrick (2021) who showed that there is a nice bijection between the set of 312-avoiding $w \in S_n$ and the set of Dyck paths of size $n$.

Ex:

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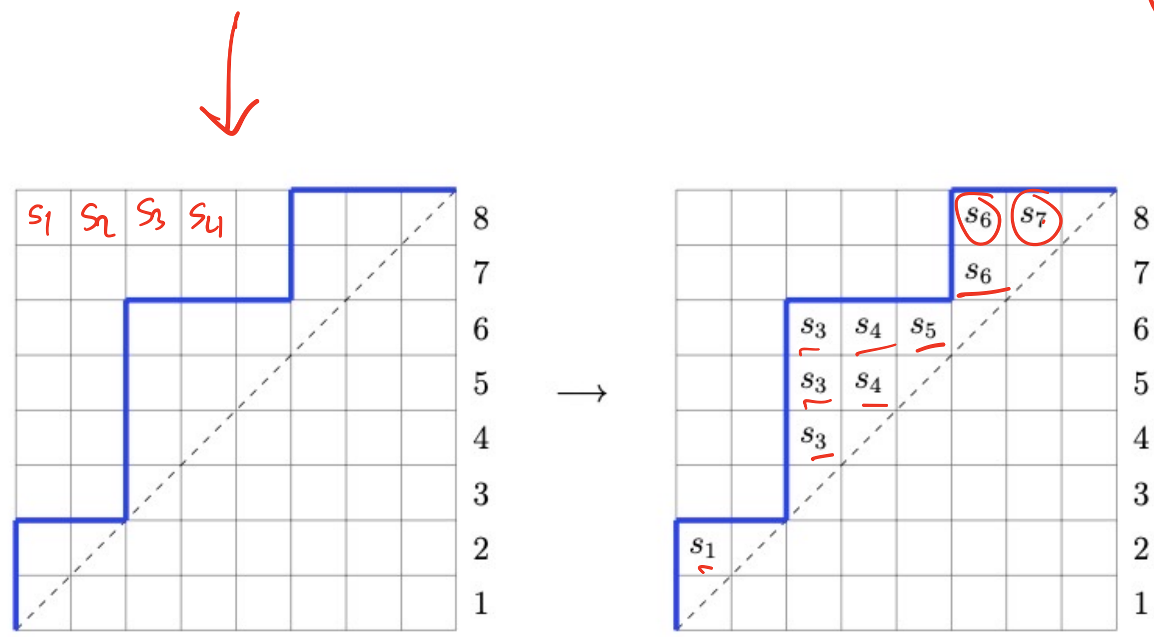



Figure 6: The construction of a permutation from a Dyck path.

$$w = s_6 s_7 s_6 \; s_3 s_4 s_5 \; s_3 s_4 \; s_3 \; s_1.$$

$$= 21654873.$$

Using this bijection and exploiting the concrete nature of Dyck paths we showed the following result:

Theorem (C.-Diaz 23):

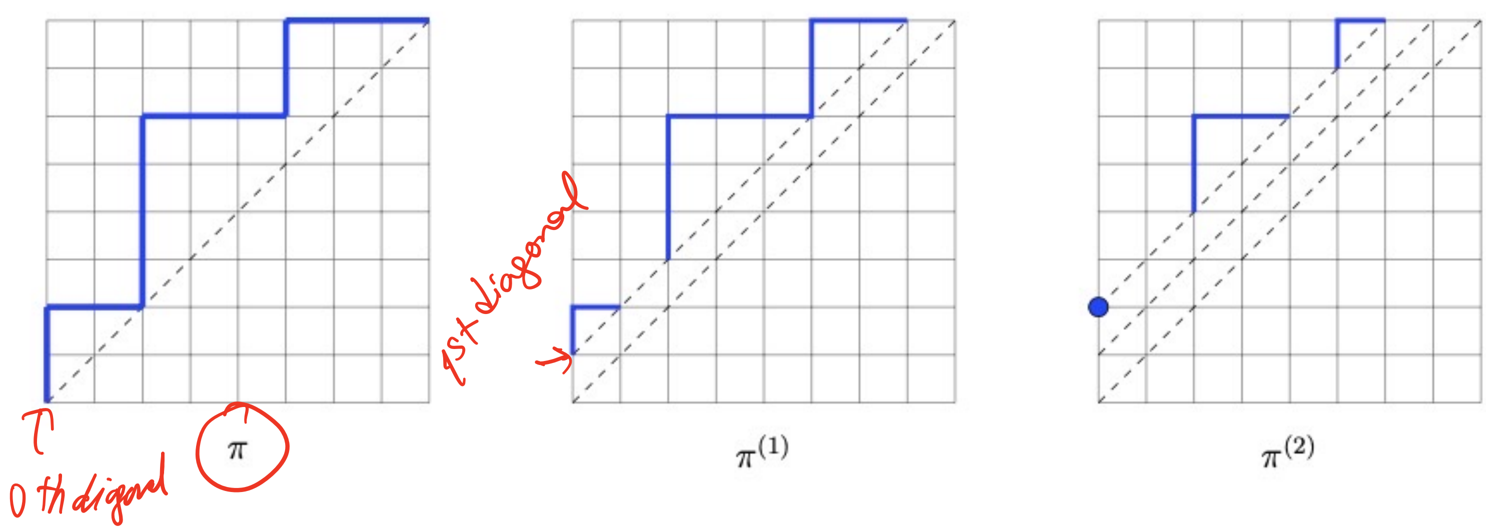
Let $NT_n^{312}$ denote the set of nearly toric varieties $X_w \subseteq Fl(n,\mathbb{C})$ such that $w$ avoids 312. Then

$$|NT_n^{312}| = (n-2)\,2^{n-3}.$$

Next, we want to see which Dyck paths correspond to the spherical Schubert varieties $X_w$, where $w$ avoids 312.

(We are not worried about $c_T(X_w) = 1$ anymore.)

**Definition:** The subpath systems of a Dyck path are defined as in the following picture:

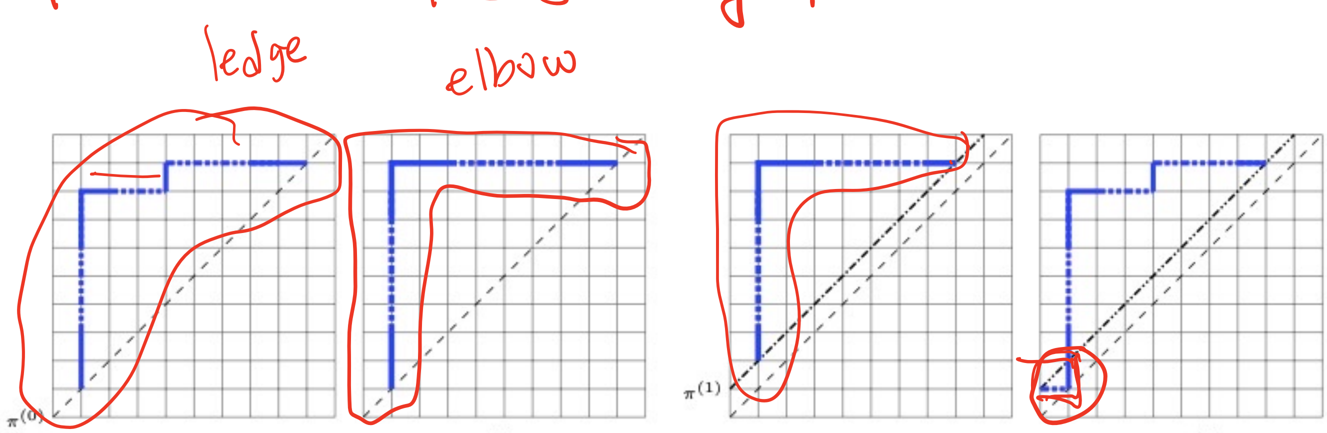



The 0th subpath system

The 1st subpath system.

The 2nd subpath system.

The ledges and elbows of $\pi^{(0)} = \pi$ and $\pi^{(1)}$ are defined as in the following picture:




(a) A ledge or an elbow of $\pi^{(0)}$.

(b) An elbow or a ledge of $\pi^{(1)}$.

Definition:

Let $\pi$ be a Dyck path of size $n$.
We say that $\pi$ is a spherical Dyck path if every connected component of $\pi^{(0)}$ is either an elbow, or a ledge as in (a) above, or every connected component of $\pi^{(1)}$ is either an elbow, or a ledge whose E-extension is the initial step of a connected component of $\pi^{(0)}$ as in (b) above.

The E-extension of NNENEE is the lattice path ENNENEE.

Theorem (C.–Diaz 2023):

Let $w$ be a 312-avoiding permutation from $S_n$. Let $\pi$ denote the Dyck path corresponding to $w$ under the Barnlow–Killpatrick bijection. Then

$X_w$ is a spherical variety

$\iff$ $\pi$ is a spherical Dyck path.

Rk: We found some other graph theoretically significant numbers while enumerating certain subfamilies of spherical Dyck paths.

THIS IS THE END UNTIL

NEXT TIME !

THANK YOU FOR TUNING IN.