Plethysm, crystals, and generating functions

Álvaro Gutiérrez (University of Bristol)

Our main problem

Let $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C}) = \{M \in \mathrm{Mat}_{2\times 2} \ : \ \mathrm{tr}(M) = 0\}$ with a Lie bracket.

Facts

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2.
3.
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Clebsch–Gordan (1800s), Kashiwara (1990s)

$\mathfrak{sl}_2$ crystals

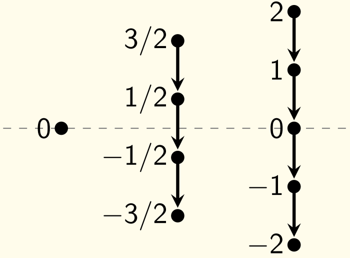
Any $\mathfrak{sl}_2$ representation V has a crystal:

A *crystal* is a vertex-weighted directed graph,

- if $x \longrightarrow y$ then $\mathrm{wt}(y) = \mathrm{wt}(x) - 1$,
- connected components are paths,
- it is weight-symmetric.

Connected components correspond to irreducible representations.

$$\mathrm{Sym}^0\,\mathbb{C}^2 \oplus \mathrm{Sym}^3\,\mathbb{C}^2 \oplus \mathrm{Sym}^4\,\mathbb{C}^2$$



A similar problem (ii)

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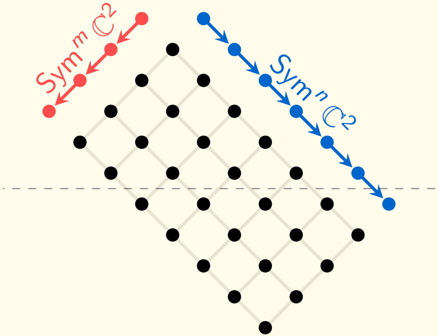

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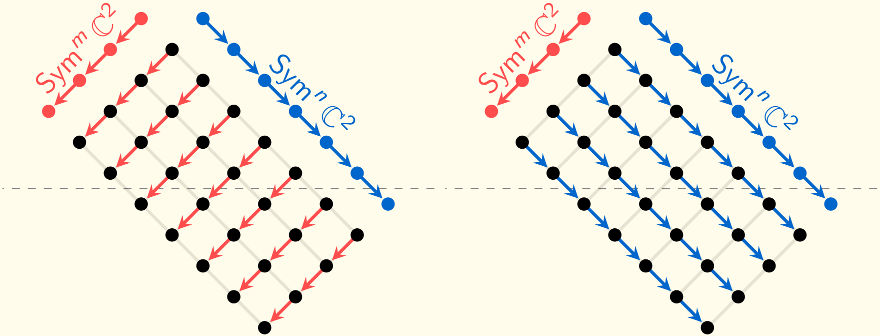

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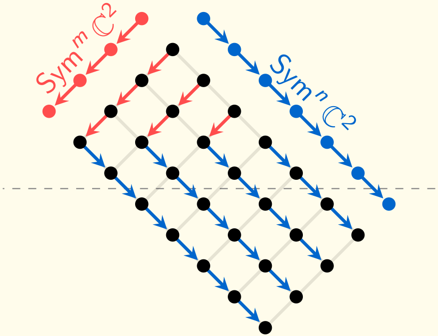

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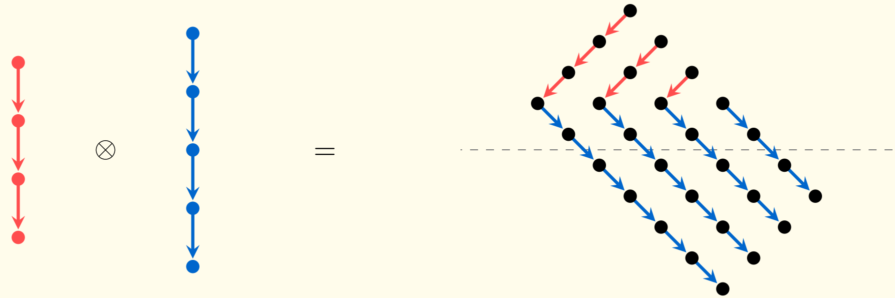

$$\mathrm{Sym}^3 \mathbb{C}^2 \otimes \mathrm{Sym}^4 \mathbb{C}^2 = \mathrm{Sym}^9 \mathbb{C}^2 \oplus \mathrm{Sym}^7 \mathbb{C}^2 \oplus \mathrm{Sym}^5 \mathbb{C}^2 \oplus \mathrm{Sym}^3 \mathbb{C}^2$$

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A canonical basis of $\operatorname{Sym}^n \operatorname{Sym}^m \mathbb{C}^2$ is in bijection with

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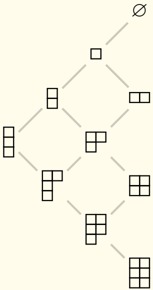
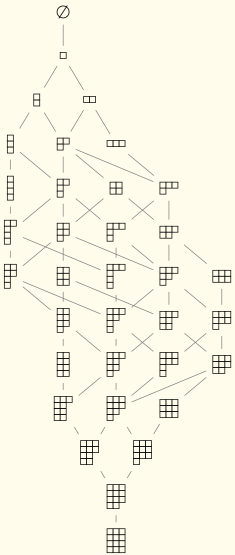
This is a ranked poset: Young's lattice.

$L(2,3) =$



Example: $L(3, 4)$



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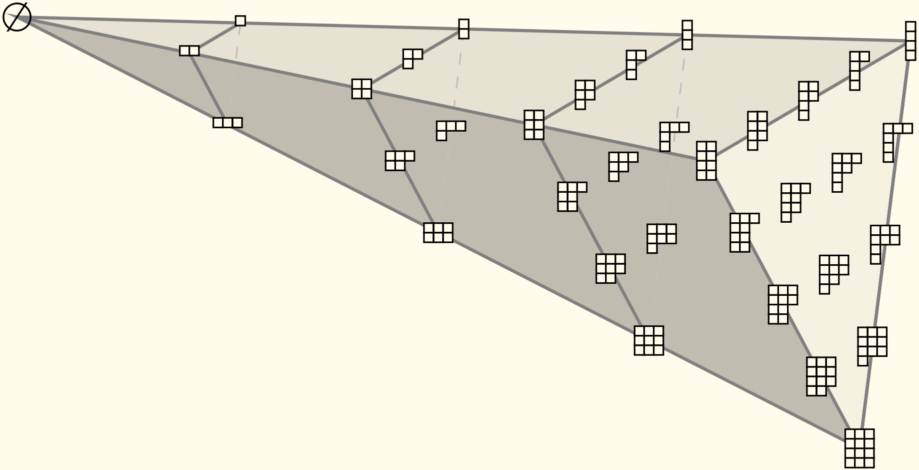

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[PDF] Unimodality of Gaussian coefficients: a constructive proof

KM O'Hara

Journal of Combinatorial Theory, Series A, 1990 • core.ac.uk

Cayley [Cay] was the first to state, and Sylvester [Syl] was the first to prove, that the coefficients of the Gaussian polynomial n

core.ac.uk

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Gutiérrez '26	•	•	•	•	•		

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$n =$	0	1	2	3	4	5	6
Rieß ‘78	•	•	•	•	•		
Stanley ‘80	•	•	•				
Lindström ‘80				•			
West ‘80					•		
Greene ‘90	•	•	•	•	•		
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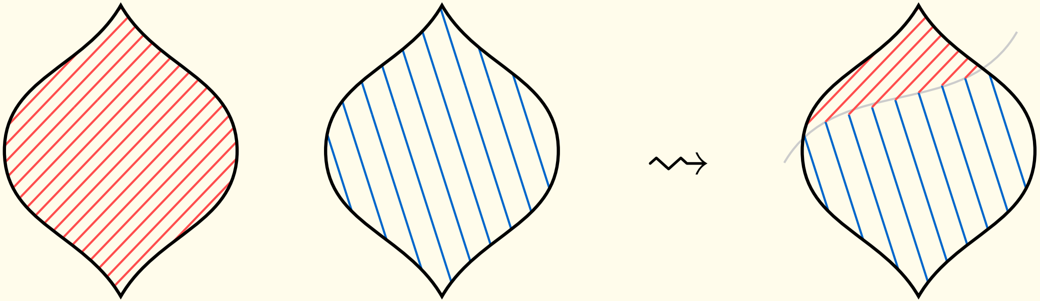

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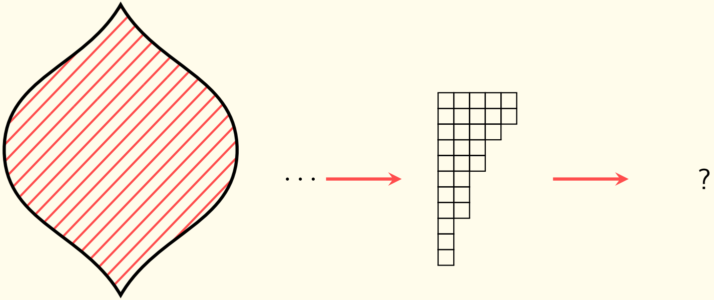

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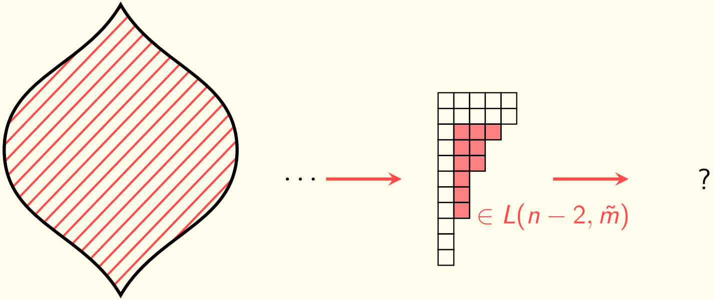

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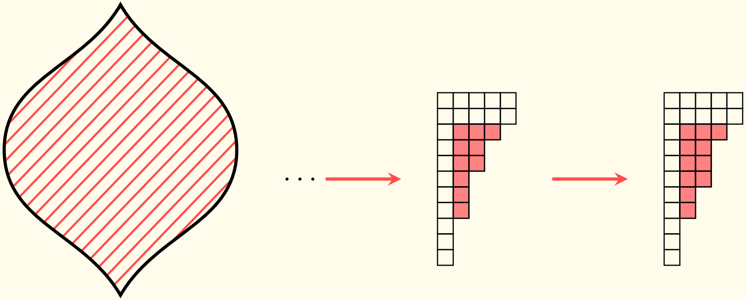

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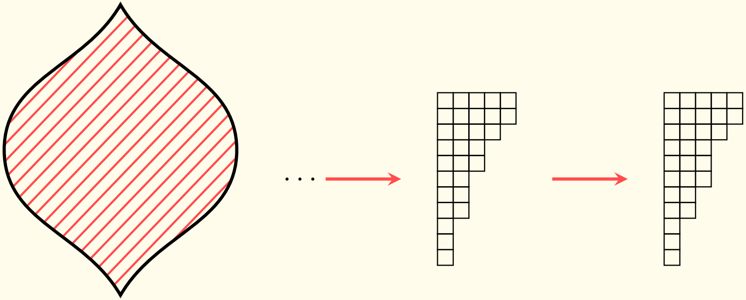

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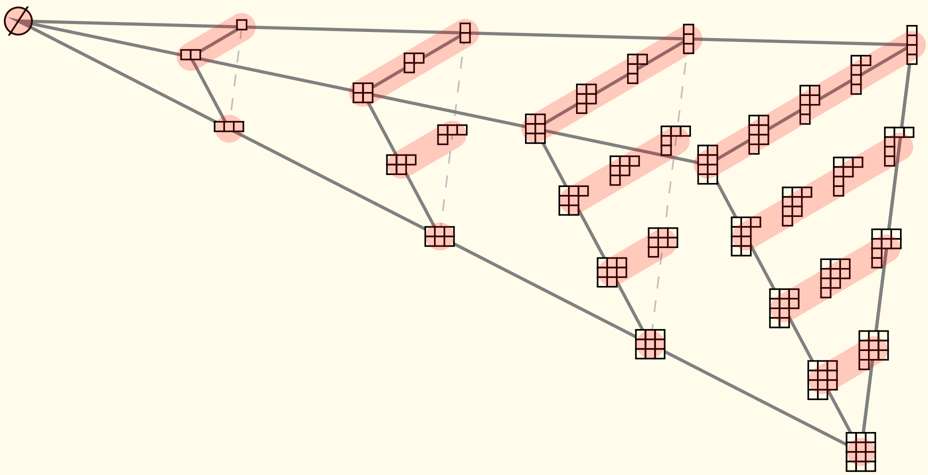

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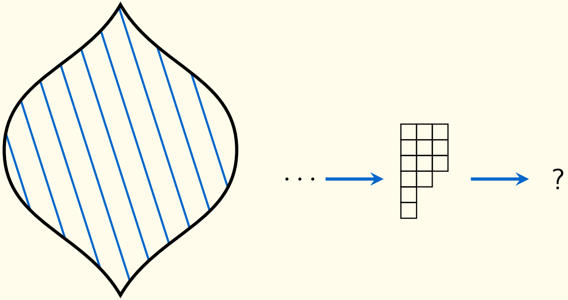

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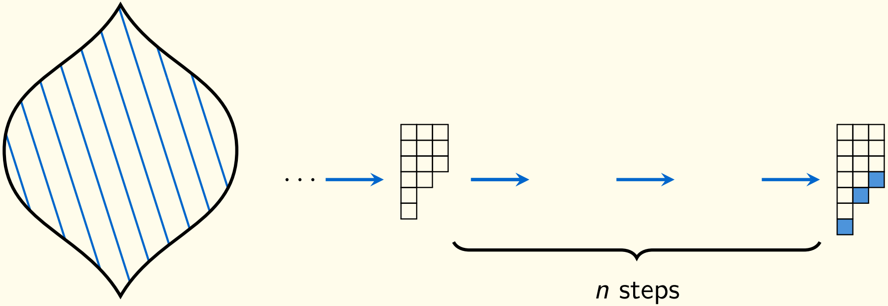

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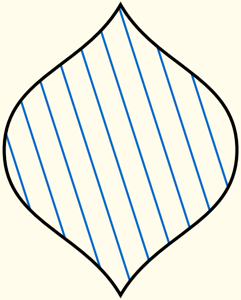

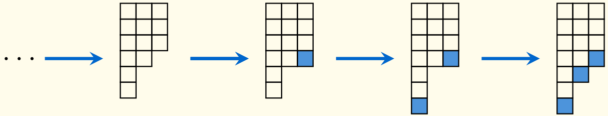

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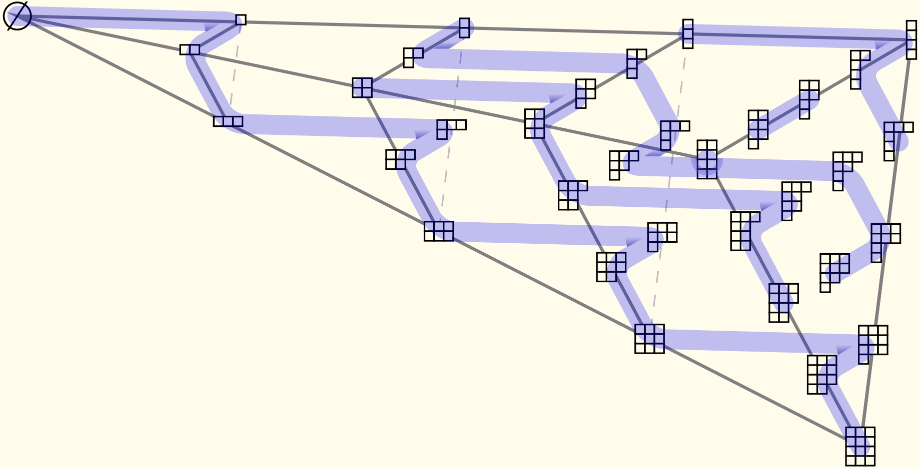

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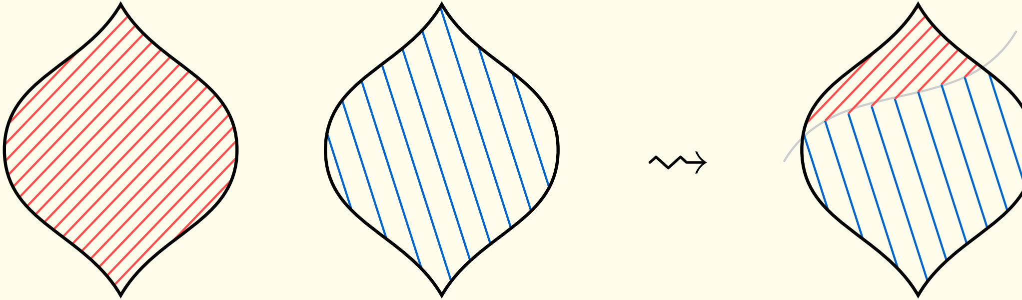

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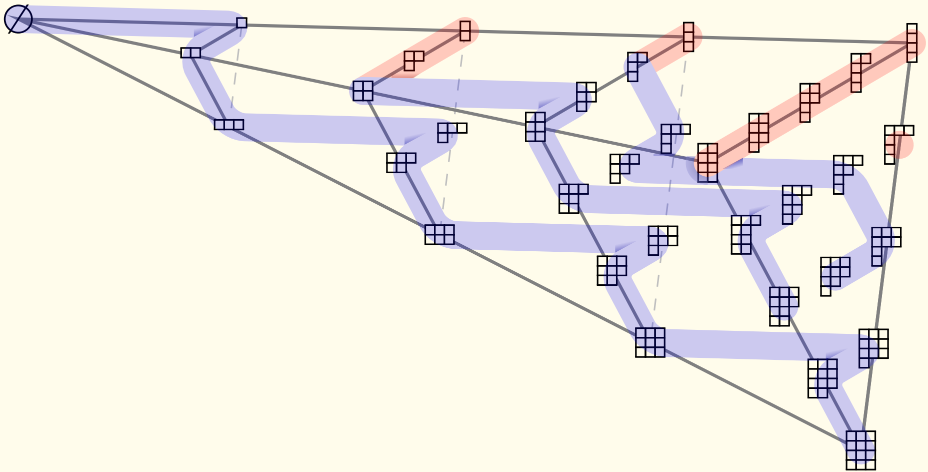

Corollaries

- Rediscover combinatorial formulas for plethystic coefficients by Orellana–Saliola–Schilling–Zabrocki'24

$$a^k_{1^3[r]} = \#\left\{ \boxed{\mathbf{c}}\boxed{\mathbf{b}}\boxed{\mathbf{a}}\,' \in \mathrm{SSYT}_2(1^3[r]) \ :\ \begin{smallmatrix} a \geq 4c+2, \\ a \neq 4c+3, \end{smallmatrix}\ b = c+1,\ \begin{smallmatrix} 3a+2b+c=k, \\ a \leq r \end{smallmatrix} \right\}.$$

$$a^k_{1^4[r]} = \#\left\{ \boxed{\mathbf{d}}\boxed{\mathbf{c}}\boxed{\mathbf{b}}\boxed{\mathbf{a}}\,' \in \mathrm{SSYT}_2(1^4[r]) \ :\ \begin{smallmatrix} a \geq b+2d+1, \\ a \neq b+2d+2, \end{smallmatrix}\ \begin{smallmatrix} c=d+1, \\ b \not\equiv c\ (2), \end{smallmatrix}\ \begin{smallmatrix} 4a+3b+2c+d=k, \\ a \leq r \end{smallmatrix} \right\}.$$

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- New recursive formulas for plethysm of Schur polys

Definition 1.1 (Plus operator). Given $f = \sum_i d_i \cdot [i]$ we let $f_{+j} = \sum_i d_i \cdot [i+j]$.

Theorem 1.2. *The character of $\Lambda^3 \mathrm{Sym}^r \mathbb{C}^2$ satisfies the recursion*

$$\begin{bmatrix} r+1 \\ 3 \end{bmatrix} = \begin{bmatrix} r \\ 3 \end{bmatrix}_{+3} + \sum [r-4k-1],$$

where the sum ranges over all $k \geq 0$ such that $4k < r - 1 - 2\delta_{r\ odd}$.

Theorem 1.3. *The character of $\Lambda^4 \mathrm{Sym}^r \mathbb{C}^2$ satisfies the following recursion:*

$$\begin{bmatrix} r+1 \\ 4 \end{bmatrix} = \begin{bmatrix} r \\ 4 \end{bmatrix}_{+4} + \sum_{k \geq 0} \begin{bmatrix} r-6k-1-3\delta_{r\ even} \\ 2 \end{bmatrix} + \sum_{k \geq 0} \begin{bmatrix} r-6k-4-3\delta_{r\ odd} \\ 2 \end{bmatrix}_{+6}.$$

Corollaries

- Rediscover combinatorial formulas for plethystic coefficients by Orellana–Saliola–Schilling–Zabrocki'24
- New combinatorial proof for a formula for the number of constituents of $\mathrm{Sym}^2 \mathrm{Sym}^n \mathbb{C}^2$, $\mathrm{Sym}^3 \mathrm{Sym}^n \mathbb{C}^2$, by Almkvist–Fossum'78
- New recursive formulas for plethysm of Schur polys

Q: What about $n = 5$ and beyond?

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Q: How to obtain these recursive formulas in general?

The formulas, again

$$\begin{array}{ccc} \mathrm{Sym}^n \mathrm{Sym}^m \mathbb{C}^2 & & \bigoplus_k (\mathrm{Sym}^{k-1} \mathbb{C}^2)^{\oplus a^k_{n[m]}} \\ \downarrow \mathrm{ch} & & \downarrow \mathrm{ch} \\ \left[\begin{smallmatrix} n+m \\ n \end{smallmatrix}\right] & & \sum_k a^k_{n[m]} \cdot [k] \end{array}$$

Theorem [G.'26]

Let $[n]_{+j} = [n+j]$. Then, for $n \leq 4$,

$$\begin{bmatrix} r+1 \\ n \end{bmatrix} = \begin{bmatrix} r \\ n \end{bmatrix}_{+n} + \sum \begin{bmatrix} \text{whatever} \\ n-2 \end{bmatrix}$$

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That is,

$$A_n(q;z) = \frac{z \cdot p(q,z)}{1 - zq^n} A_{n-2}(q;z).$$

Theorem [G.–Orellana–Saliola–Schilling–Zabrocki'25$^+$]

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Let

$$\mathbb{A}_\mu(x_1, \ldots, x_m; y_1, \ldots, y_m) = \sum_\nu \sum_\lambda a^\lambda_{\mu[\nu]} x^\nu y^\lambda,$$

where $a^\lambda_{\mu[\nu]}$ is the coefficient of s_λ in $s_\mu[s_\nu]$.

Theorem [G.–Orellana–Saliola–Schilling–Zabrocki'25^+]

$\mathbb{A}_\mu(q; z)$ is a rational function.
In particular, plethysm coefficients satisfy linear recurrences.