A reflection principle for nonintersecting paths and lozenge tilings with free boundaries

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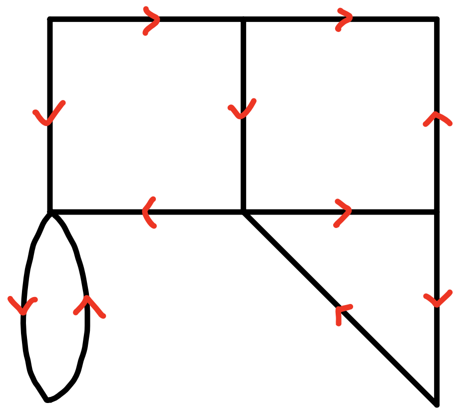
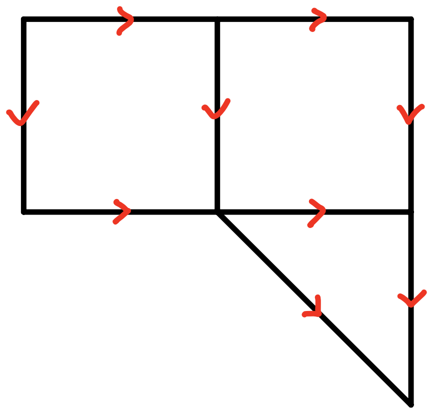
A locally finite and acyclic directed graph

In this talk, every graph will be a Locally Finite and Acyclic Directed Graph.

- Directed Graph: every edge is directed.
- Acyclic Directed Graph: a directed graph with no directed cycle.
- Locally Finite Graph: every vertex has finite degree.





A directed paths on an acyclic directed graph

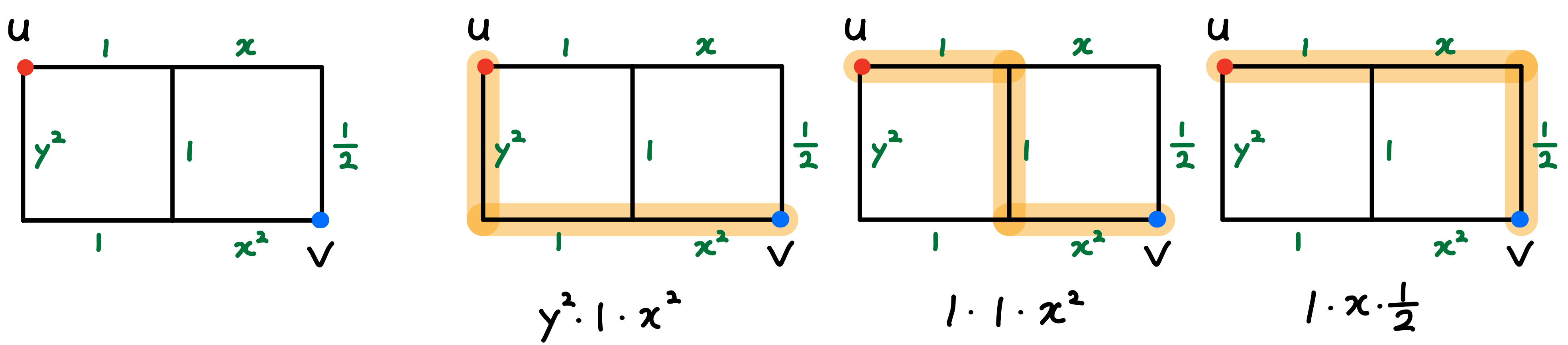
Given a graph G, pick two points u and v on the graph.

We assume that each edge of G is weighted (by a number or a monomial).

Given a directed path P from u to v, the weight of P, denoted by $\mathrm{wt}(P)$, is the product of weight of all edges that constitute the path.

Let $\mathscr{P}(u,v)$ be the set of all directed paths from u to v and $\mathrm{GF}[\mathscr{P}(u,v)]$ be the sum of weight of all paths in $\mathscr{P}(u,v)$, i.e.

$$\mathrm{GF}[\mathscr{P}(u,v)] = \sum_{P \in \mathscr{P}(u,v)} \mathrm{wt}(P).$$



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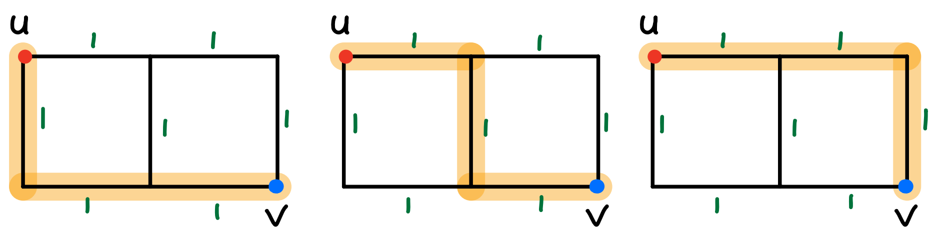
Note: If all edges are weighted by 1,
then $\mathrm{GF}[\mathscr{P}(u,v)] =$(number of directed paths from u to v).



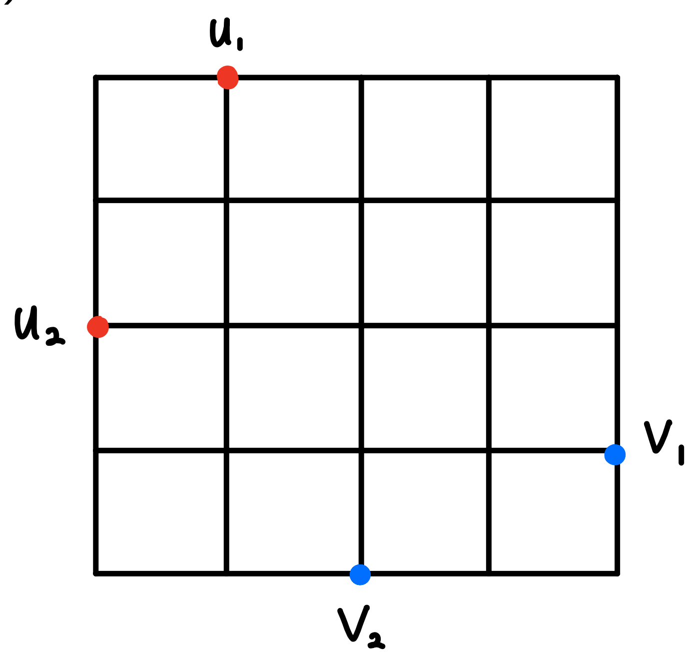
Families of paths

Let us generalized the setting from the previous slide.

Let $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_m)$ be m-tuples of points on G.

If P_i is a path from u_i to v_i for $i = 1, \ldots, m$,
then (the weight of the m-tuple $(P_1, \ldots, P_m)$) $:= \prod_{i \in [m]} \mathrm{wt}(P_i)$.

Let $\mathscr{P}(\mathbf{u}, \mathbf{v})$ be the set of all m-tuples of paths $(P_1, \ldots, P_m)$ such that $P_i \in \mathscr{P}(u_i, v_i)$ for $i = 1, \ldots, m$ and $\mathrm{GF}[\mathscr{P}(\mathbf{u}, \mathbf{v})]$ be the sum of weight of all m-tuples in $\mathscr{P}(\mathbf{u}, \mathbf{v})$.



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Observation

$\mathrm{GF}[\mathscr{P}(\mathbf{u}, \mathbf{v})] = \prod_{i \in [m]} \mathrm{GF}[\mathscr{P}(u_i, v_i)]$.

This is because paths P_i and P_j do not interact for all $i \neq j$!

Families of nonintersecting paths

We now consider a subset of $\mathscr{P}(\mathbf{u}, \mathbf{v})$.

m-tuples of paths $(P_1, \ldots, P_m)$ are *nonintersecting* if P_i and P_j do not intersect for all $i \neq j$.

Let $\mathscr{P}_0(\mathbf{u}, \mathbf{v})$ be the set of all m-tuples of paths $(P_1, \ldots, P_m)$ such that $P_i \in \mathscr{P}(u_i, v_i)$ for $i = 1, \ldots, m$ and P_i and P_j do not intersect for all $i \neq j$.

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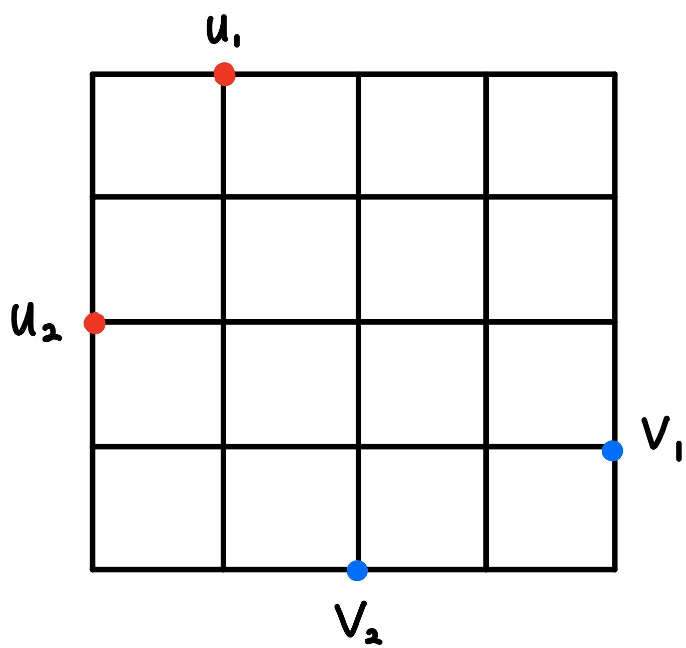

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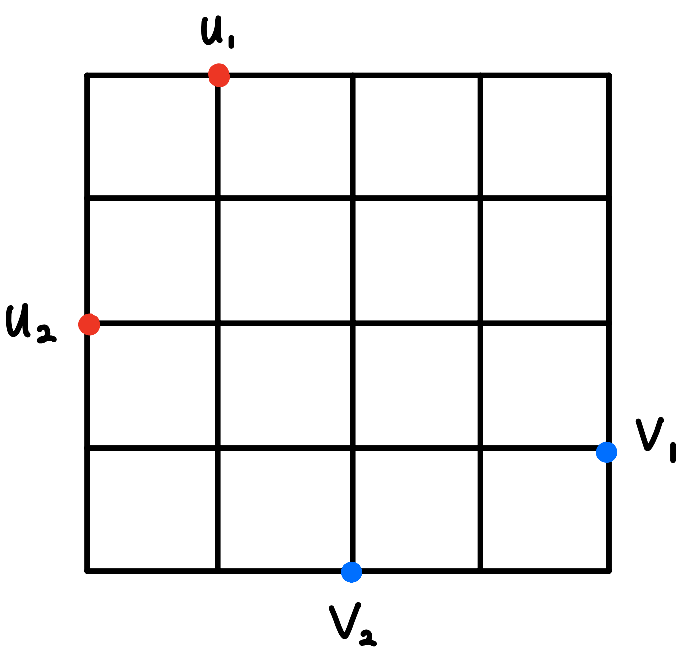
$\mathrm{GF}[\mathscr{P}_0(\mathbf{u}, \mathbf{v})]$ be the sum of weight of all m-tuples in $\mathscr{P}_0(\mathbf{u}, \mathbf{v})$.

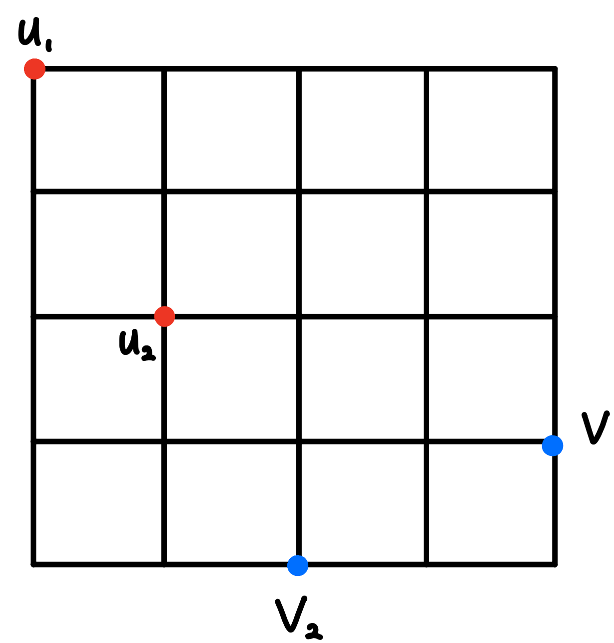
Q: What is $\mathrm{GF}[\mathscr{P}_0(\mathbf{u}, \mathbf{v})]$?

Compatibility condition on starting/ending points

The question in the previous slide has a simple answer if m-tuples of starting and ending points are "compatible".

We say that **u** and **v** are *compatible* if for $i, j, k, l \in [m]$ such that $i < j$ and $k < l$, every path $P \in \mathscr{P}(u_i, v_l)$ intersects with every path $Q \in \mathscr{P}(u_j, v_k)$.





Lindström–Gessel–Viennot theorem

Let $\mathrm{M}(\mathbf{u}, \mathbf{v})$ be the $m \times m$ matrix defined by

$$\mathrm{M}(\mathbf{u}, \mathbf{v}) = \Big[\mathrm{GF}[\mathscr{P}(u_i, v_j)] \Big]_{1 \le i,j \le m}$$

and we call this matrix the *path matrix from* $\mathbf{u}$ *to* $\mathbf{v}$.

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Theorem (Lindström–Gessel–Viennot)

If **u** and **v** are compatible, then

$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u}, \mathbf{v})] = \det \mathrm{M}(\mathbf{u}, \mathbf{v}).$$

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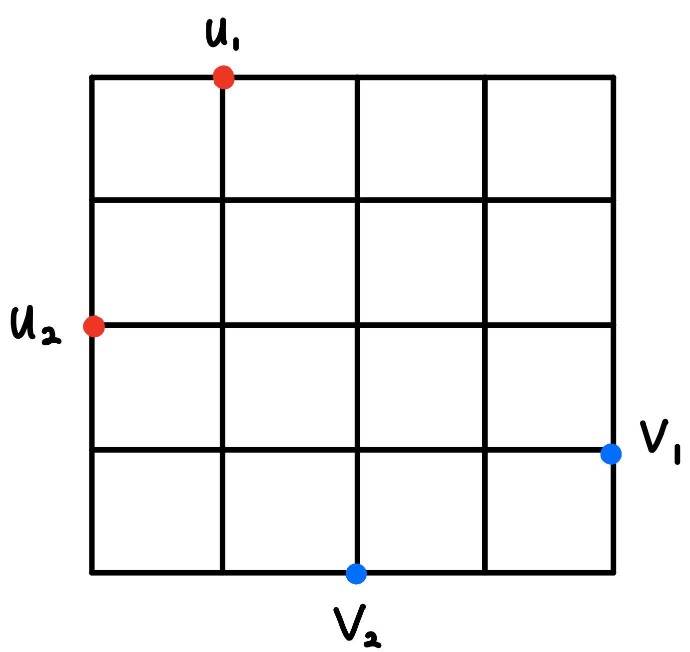
If **u** and **v** are compatible, then

$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u}, \mathbf{v})] = \det \mathrm{M}(\mathbf{u}, \mathbf{v}).$$

Note: When **u** and **v** are not compatible, the determinant give "signed" enumeration of m-tuples of nonintersecting paths (we will not discuss this case in this talk.)



$$\det M(u,v) = \begin{vmatrix} GF(u_1,v_1) & GF(u_1,v_2) \\ GF(u_2,v_1) & GF(u_2,v_2) \end{vmatrix}$$

$$= \begin{vmatrix} \binom{6}{3} & \binom{5}{4} \\ \binom{5}{1} & \binom{4}{2} \end{vmatrix}$$

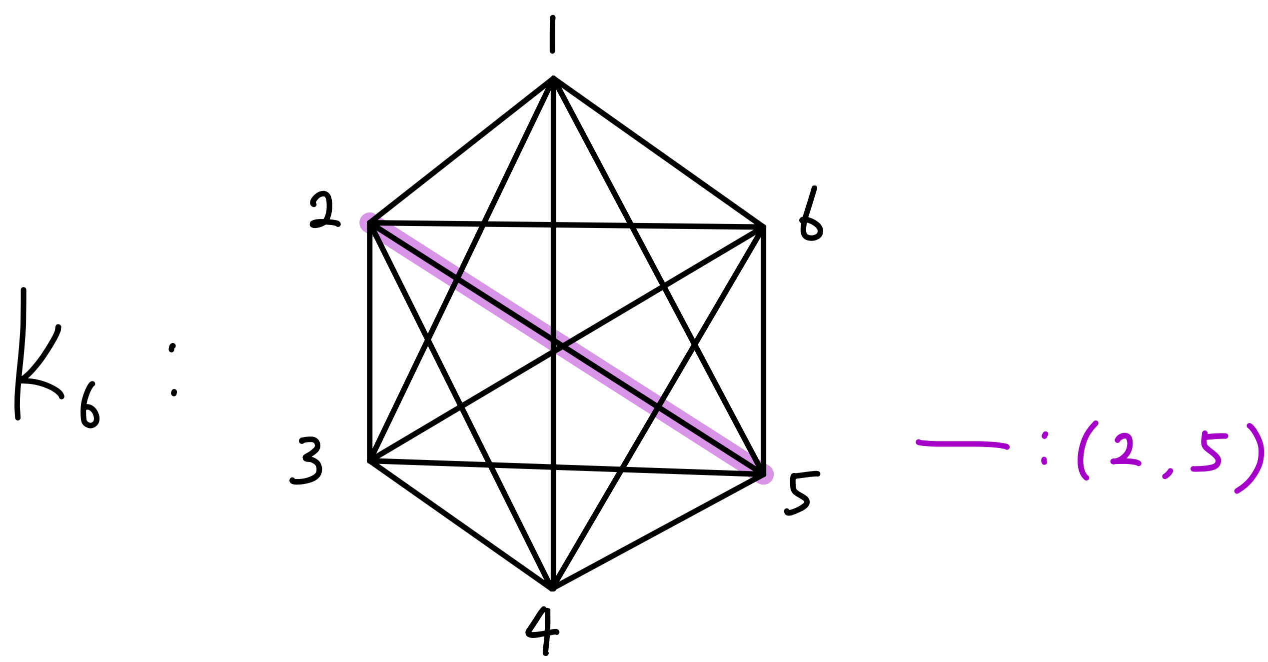
$$= \begin{vmatrix} 20 & 5 \\ 5 & 6 \end{vmatrix}$$

$$= 120 - 25 = 95.$$

Pfaffian (1/2)

A $2m \times 2m$ matrix $A = [a_{i,j}]_{1 \leq i,j \leq 2m}$ is *skew symmetric* if $a_{i,j} + a_{j,i} = 0$ for all $i, j \in [2m]$.

We now define a *Pfaffian* of a skew-symmetric matrix A, denoted by $\mathrm{Pf}_{2m}(A)$ or $\mathrm{Pf}(A)$.



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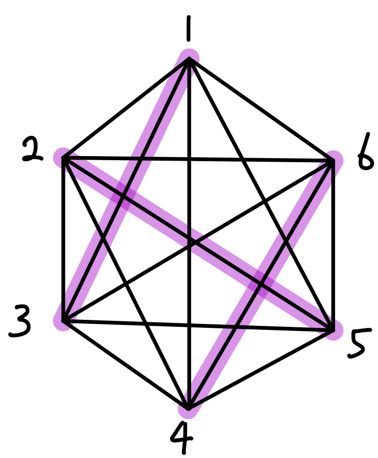
Consider a complete graph K_{2m} and label its vertices by the elements of $[2m] = \{1, \ldots, 2m\}$.

A *1-factor* (or a *perfect matching*) of K_{2m} is a collection of m edges of K_{2m} that covers every vertex exactly once.

Let $\mathscr{F}_m$ be the set of 1-factors of K_{2m}.



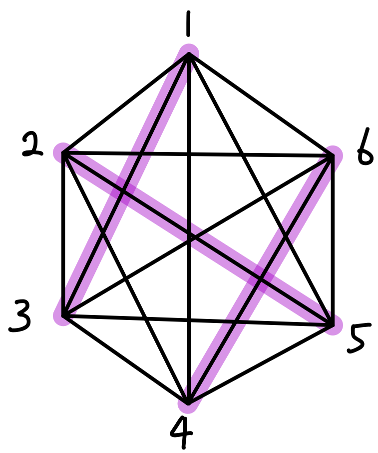
: $\pi = (1,3), (2,5), (4,6)$

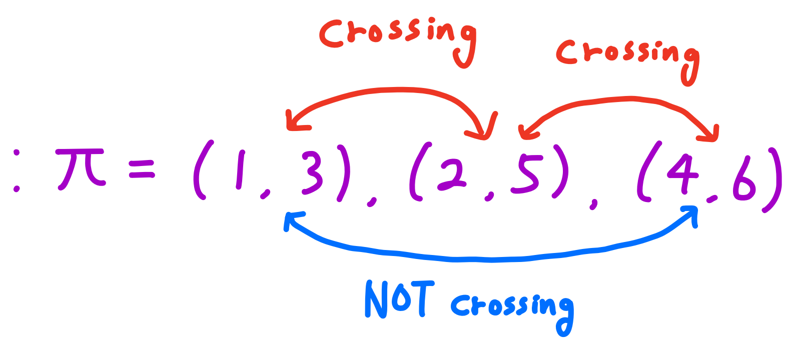
$* \; |\mathcal{F}_3| = 15.$

Two edges (i, j) and (k, l) of $\pi \in \mathscr{F}_m$ are *crossing* if either $i < k < j < l$ or $k < i < l < j$ holds.

The crossing number of π, denoted by $\mathrm{cr}(\pi)$, is the number of crossed pairs in π.

The sign of the 1-factor $\pi \in \mathscr{F}_m$, denoted by $\mathrm{sgn}(\pi)$, is $\mathrm{sgn}(\pi) := (-1)^{\mathrm{cr}(\pi)}$.





$\mathrm{cr}(\pi) = 2.$

$\mathrm{sgn}(\pi) = (-1)^{\mathrm{cr}(\pi)} = (-1)^2 = 1.$

Pfaffian (2/2)

Two edges (i, j) and (k, l) of $\pi \in \mathscr{F}_m$ are *crossing* if either $i < k < j < l$ or $k < i < l < j$ holds.

The crossing number of π, denoted by $\mathrm{cr}(\pi)$, is the number of crossed pairs in π.

The sign of the 1-factor $\pi \in \mathscr{F}_m$, denoted by $\mathrm{sgn}(\pi)$, is $\mathrm{sgn}(\pi) := (-1)^{\mathrm{cr}(\pi)}$.

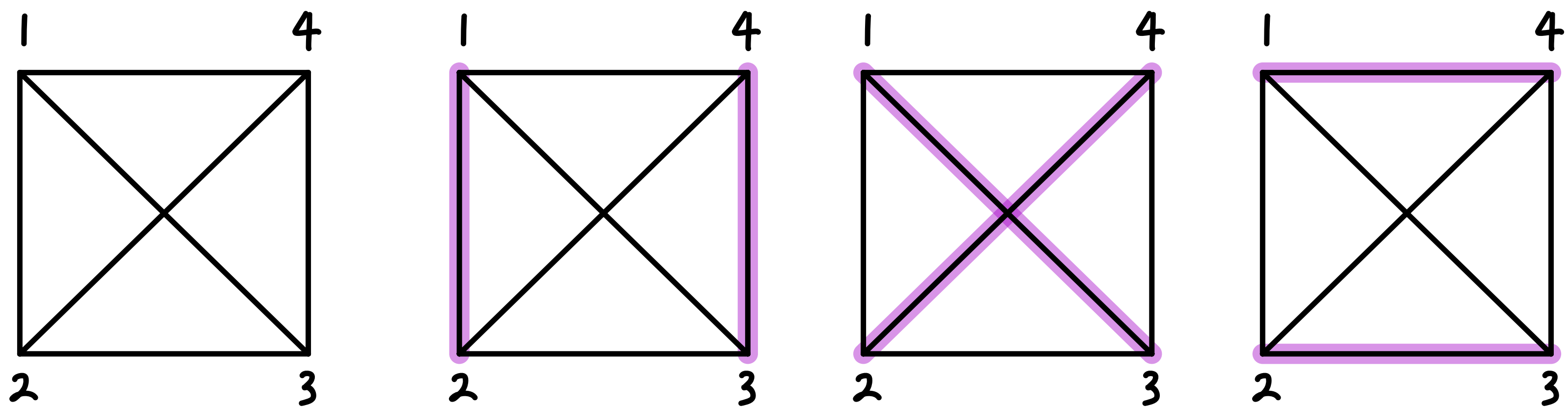
The Pfaffian of a skew-symmetric matrix $A = [a_{i,j}]_{1 \le i,j \le 2m}$ is defined as follows.

$$\mathrm{Pf}(A) = \sum_{\pi \in \mathscr{F}_m} \mathrm{sgn}(\pi) \prod_{(i,j) \in \pi} a_{i,j}.$$

Pfaffian of a 4 × 4 matrix

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$$

$K_4 =$



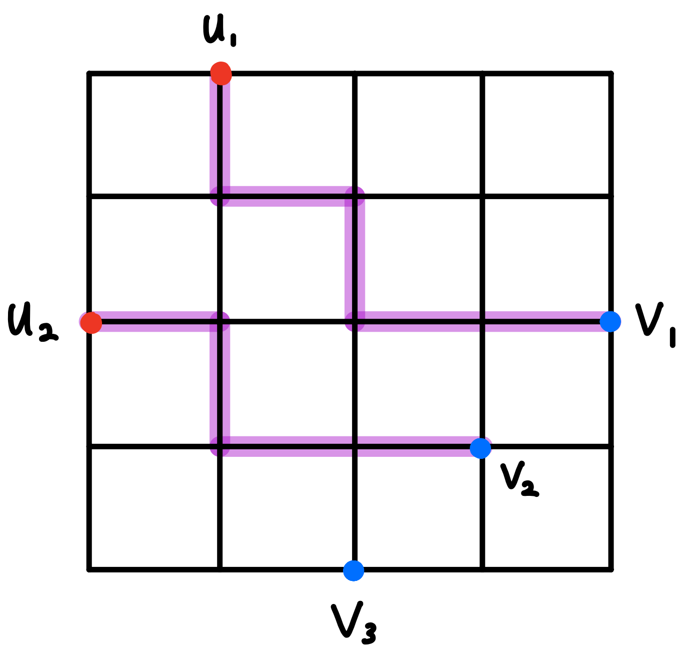
$$Pf(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

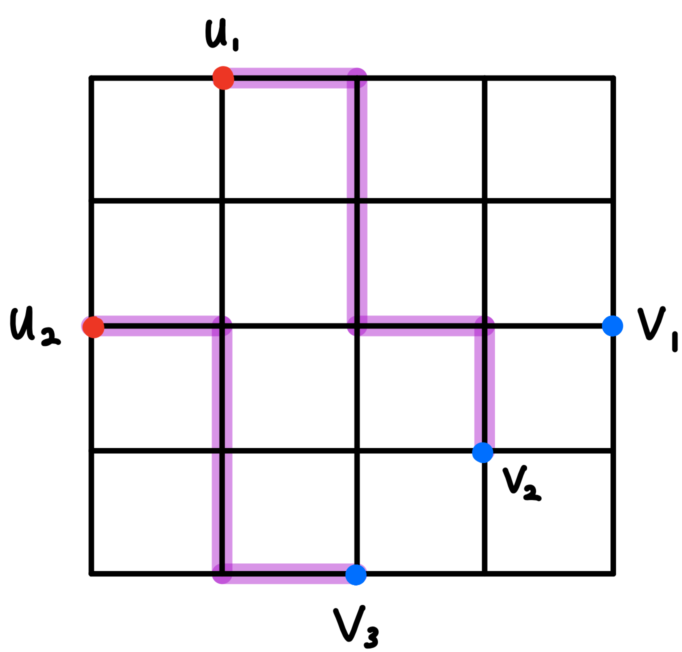
Families of paths with more ending points

Let $\mathbf{u} = (u_1, \ldots, u_m)$ be an m-tuple and $\mathbf{v} = (v_1, \ldots, v_n)$ be an n-tuple of points on G, where $n \geq m$. Assume that $\mathbf{u}$ and $\mathbf{v}$ are compatible.

Let $\mathscr{P}_0(\mathbf{u}, \mathbf{v})$ be the set of all nonintersecting m-tuples of paths $(P_1, \ldots, P_m)$ such that $P_i \in \mathscr{P}(u_i, v_{k_i})$ for $i = 1, \ldots, m$ and $1 \leq k_1 \leq \cdots \leq k_m \leq n$.

Let $\mathrm{GF}[\mathscr{P}_0(\mathbf{u}, \mathbf{v})]$ be the sum of weight of all m-tuples in $\mathscr{P}_0(\mathbf{u}, \mathbf{v})$.





Okada–Stembridge theorem

Theorem (Okada–Stembridge)

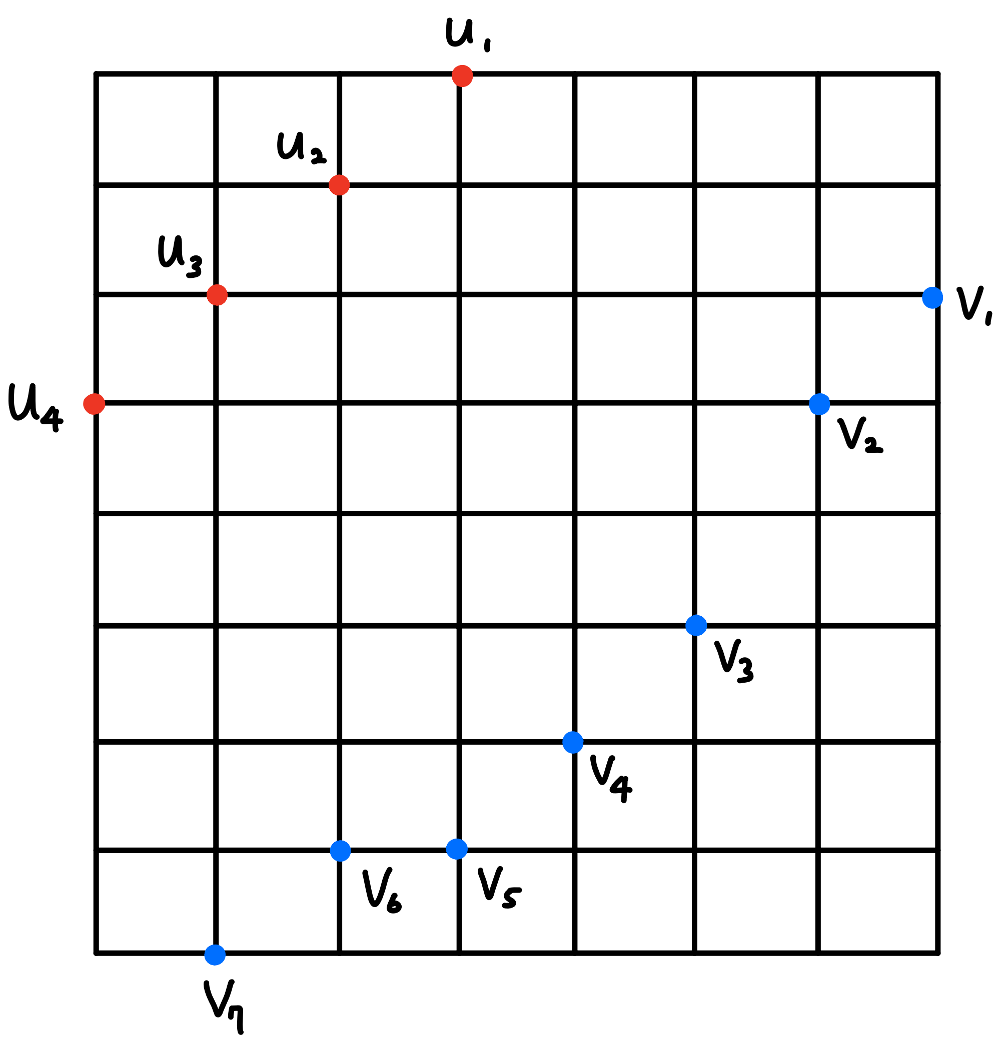
If m is even, then

$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u},\mathbf{v})] = \mathrm{Pf}\left[\,\mathrm{Q}(\mathbf{u},\mathbf{v})\right],$$

where $\mathrm{Q}(\mathbf{u},\mathbf{v}) = \left[Q_{i,j}\right]_{1\le i,j\le m}$ is the skew-symmetric matrix defined by

$$Q_{i,j} = \sum_{s<t}\left[\,\mathrm{GF}[\mathscr{P}(u_i,v_s)]\cdot\mathrm{GF}[\mathscr{P}(u_j,v_t)] - \mathrm{GF}[\mathscr{P}(u_i,v_t)]\cdot\mathrm{GF}[\mathscr{P}(u_j,v_s)]\right].$$

An example for Okada–Stembridge theorem

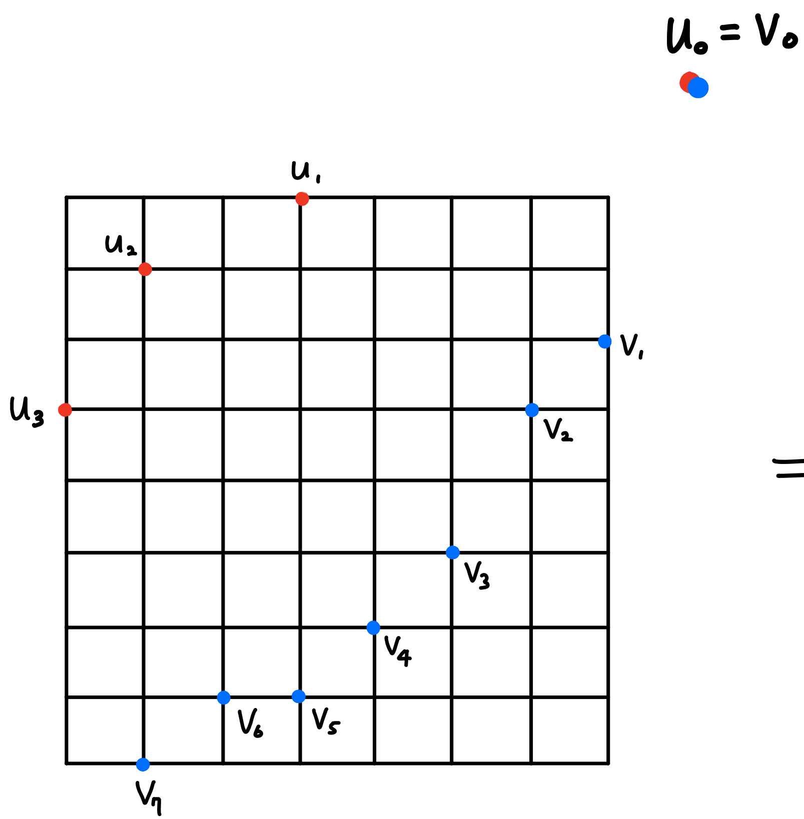


$$Q(u,v) = \begin{pmatrix} 0 & Q_{1,2} & Q_{1,3} & Q_{1,4} \\ -Q_{1,2} & 0 & Q_{2,3} & Q_{2,4} \\ -Q_{1,3} & -Q_{2,3} & 0 & Q_{3,4} \\ -Q_{1,4} & -Q_{2,4} & -Q_{3,4} & 0 \end{pmatrix}$$

$$Q_{1,2} = \left[\binom{6}{2} \cdot \binom{6}{2} - \binom{6}{3} \cdot \binom{6}{1} \right] + \left[\binom{6}{2} \cdot \binom{7}{4} - \binom{7}{5} \cdot \binom{6}{1} \right] + \cdots.$$

Case when m is odd: adding a “Phantom” vertex $u_0 = v_0$

Stembridge pointed out that this theorem can also be applied when m is odd by adding a “Phantom” vertex $u_0 = v_0$:



Let $u' = (u_0, u_1, \cdots, u_m)$ and $v' = (v_0, v_1, \cdots, v_n)$.

($m+1$ — even!)

$\Rightarrow$ Then $GF\left[\mathcal{P}_0(u,v)\right] = GF\left[\mathcal{P}_0(u',v')\right]$

can apply Okada–Stembridge!

In this case, the resulting Pfaffian has order $m + 1$.

A natural question

While there is no parity condition on m (the number of starting points) in Lindström–Gessel–Viennot theorem, m should be even to apply Okada-Stembridge theorem directly.

One can still apply Okada–Stembridge's theorem when m is odd, following Stembridge's remark, but the Pfaffian has a different order than when m is even.

Q: Is there an alternative formula that does not depend on the parity of m?

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Q: Is there an alternative formula that does not depend on the parity of m?

A: Yes.

A new theorem

Theorem (Byun, 2025)

We have

$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u},\mathbf{v})]^2 = \det\left[\mathrm{M}(\mathbf{u},\mathbf{v})U_n\,\mathrm{M}(\mathbf{u},\mathbf{v})^T\right] = \det\left[\mathrm{M}(\mathbf{u},\mathbf{v})U_n^T\,\mathrm{M}(\mathbf{u},\mathbf{v})^T\right],$$

where $\mathrm{M}(\mathbf{u},\mathbf{v})$ is the path matrix and $U_n = [u_{i,j}]_{1\le i,j\le n}$ is the upper

↳ $m \times n$ matrix!

triangular matrix defined by $u_{i,j} = \begin{cases} 2, & \text{if } i < j \\ 1, & \text{if } i = j \\ 0, & \text{if } i > j \end{cases}.$

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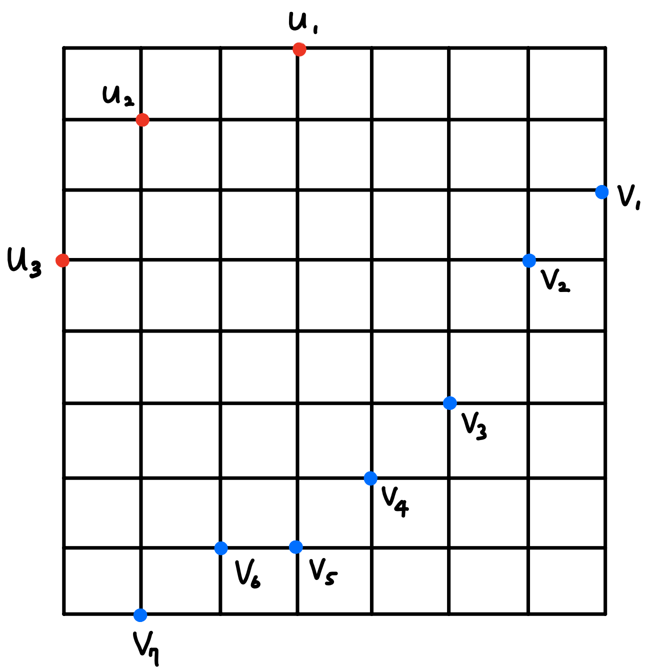
where $\mathrm{M}(\mathbf{u},\mathbf{v})$ is the path matrix and $U_n = [u_{i,j}]_{1\le i,j\le n}$ is the upper triangular matrix defined by $u_{i,j} = \begin{cases} 2, & \text{if } i<j \\ 1, & \text{if } i=j \\ 0, & \text{if } i>j \end{cases}$.

Note: there is no parity condition on m!

Main ingredients of Proof: (Lindström–Gessel–Viennot theorem) and (Okada's result on the sum of maximum minors of a matrix).

An example for the new theorem



$$M(u,v) = \begin{bmatrix} \binom{6}{2} & \binom{6}{3} & \binom{7}{5} & \binom{7}{6} & \binom{7}{7} & 0 & 0 \\ \binom{7}{1} & \binom{7}{2} & \binom{8}{4} & \binom{8}{5} & \binom{8}{6} & \binom{7}{6} & \binom{7}{7} \\ 0 & \binom{6}{0} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{6}{4} & \binom{6}{5} \end{bmatrix}$$

$$U_7 = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{M(u,v)\, U_7\, M(u,v)^T} = \begin{bmatrix} 4096 & 18769 & 13259 \\ 5551 & 36100 & 31100 \\ 1205 & 11840 & 12769 \end{bmatrix}$$

$$\det[\quad] = 47527052049 = 218007^2.$$

Hence, $GF[\mathcal{P}_0(u,v)] = 218007.$

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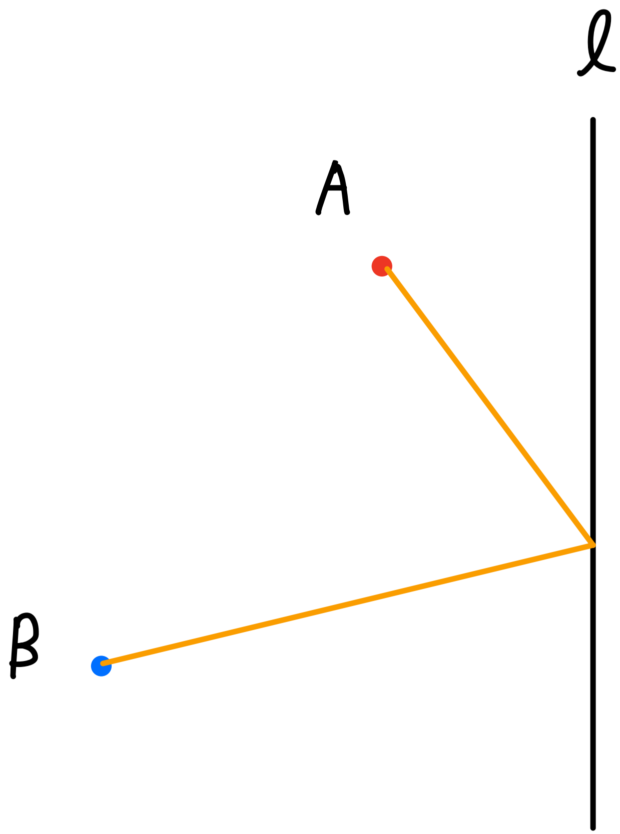
"Reflection" in Mathematics (1/2)

Consider a line l on $\mathbb{R}^2$ and two points A and B that lie on the same side with respect to l.

Q: What is the shortest distance of a path P joining A ant B that touches l?



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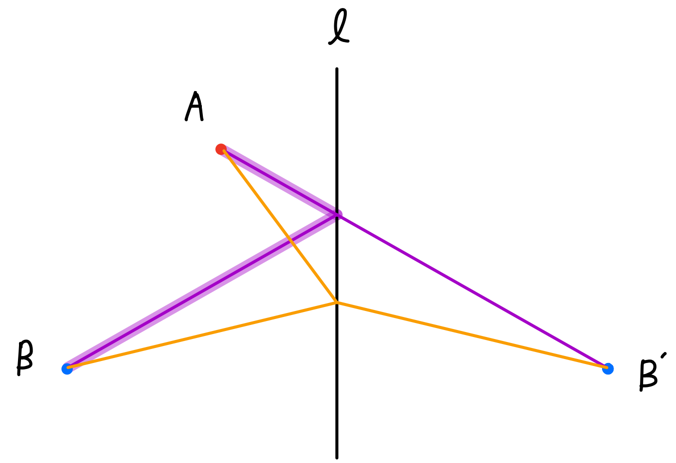
Q: What is the shortest distance of a path P joining A ant B that touches l?

A: Reflect B across l and call it B'. Let $P' = \overline{AB'}$. Then P' must go through l and P is obtained from P' by reflecting it across l.

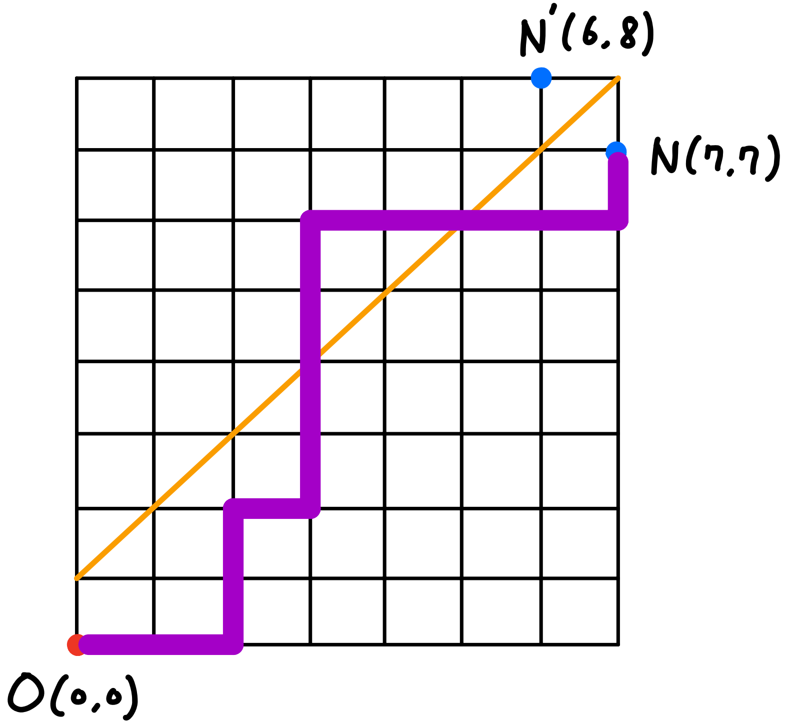
Note: By reflecting B, we could instead solve an equivalent problem (finding shortest path joining A and B') that has no further restriction.

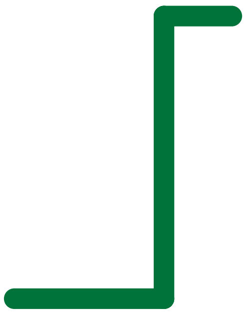


"Reflection" in Mathematics (2/2)

Consider a two points $O = (0, 0)$ and $N = (n, n)$ on $\mathbb{Z}^2$ (assume $n > 1$). Assume that every edge in $\mathbb{Z}^2$ is oriented directed toward east or north.

Q: How many (directed) paths are there that 1) join O and N and 2) go through $y = x$ at least once?





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Q: How many (directed) paths are there that 1) join O and N and 2) go through $y = x$ at least once?

A: (André's reflection principle) Using reflection, one can show that the number of such paths is the same as the number of path joining $O = (0, 0)$ and $N' = (n - 1, n + 1)$ (with no further restriction). Hence the answer is $\binom{2n}{n-1}$.

Note: By reflecting the path, one could instead solve an equivalent problem (finding the number of paths joining O and N') that has no further restriction.

Common theme of the two examples

: by reflecting some objects (points, lattice paths), one can convert original (optimization, enumeration) problems that have some restrictions into new problems that do not have any restriction.

$\Rightarrow$ The problems become simpler!

Reflection principles in Mathematics

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Digression: There are more "Reflection principles" in Mathematics: set theory, probability, complex analysis...

Reflection principles in Mathematics

Common theme of the two examples

: by reflecting some objects (points, lattice paths), one can convert original (optimization, enumeration) problems that have some restrictions into new problems that do not have any restriction.

$\Rightarrow$ The problems become simpler!

Digression: There are more "Reflection principles" in Mathematics: set theory, probability, complex analysis...

We add one more reflection principle
: a reflection principle for nonintersecting paths!

A combinatorial interpretation of the new result

We can give a combinatorial interpretation of the new result, which we call "A reflection principle for nonintersecting path."

This is obtained by interpreting $\mathrm{M}(\mathbf{u}, \mathbf{v})$, U_n (or U_n^T), and $\mathrm{M}(\mathbf{u}, \mathbf{v})^T$ as path matrices of certain acyclic directed graphs.

A combinatorial interpretation of the new result

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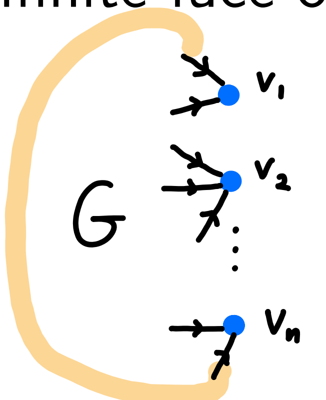
This is obtained by interpreting $\mathrm{M}(\mathbf{u}, \mathbf{v})$, U_n (or U_n^T), and $\mathrm{M}(\mathbf{u}, \mathbf{v})^T$ as path matrices of certain acyclic directed graphs.

Consider a locally finite and acyclic directed graph G and an m-tuple $\mathbf{u} = (u_1, \ldots, u_m)$ and an n-tuple $\mathbf{v} = (v_1, \ldots, v_n)$ of vertices on G, where $m \leq n$.

Assume further that
1) the vertices $v_1, \ldots, v_n$ are sinks (that is, there are no outgoing edges from these n vertices in G) and
2) these n vertices are on the infinite face of G in a cyclic order.

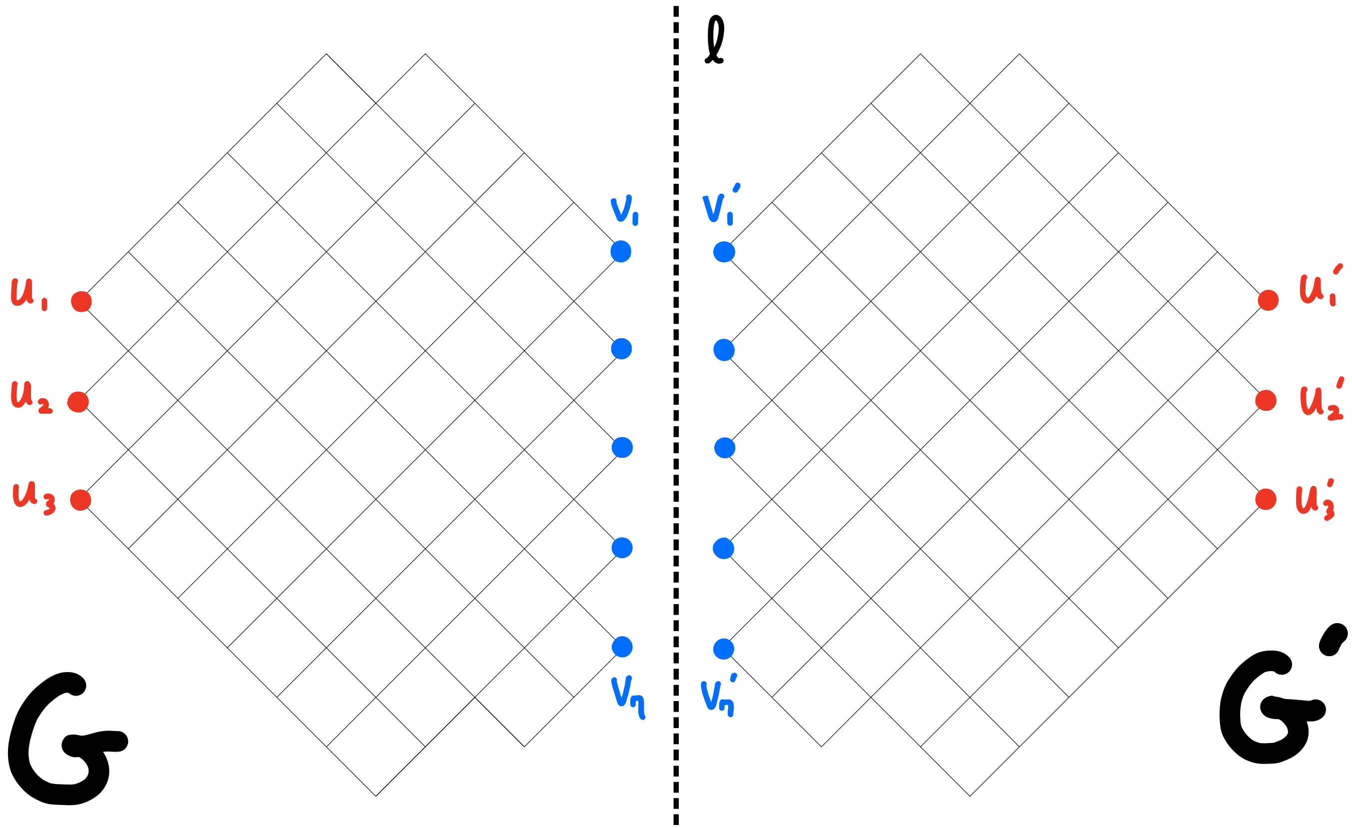


A combinatorial interpretation of the new result (continue)

Reflect G across a "good" line l and call the mirror image G'. We reverse the orientation of all edges of G' and keep the weight on the edges.

Denote the mirror images of $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ by $\mathbf{u}' = (u'_1, \ldots, u'_m)$ and $\mathbf{v}' = (v'_1, \ldots, v'_n)$, respectively, and $G_{sym} := G \cup G'$.

Note that $v'_1, \ldots, v'_n$ are sources of G'.



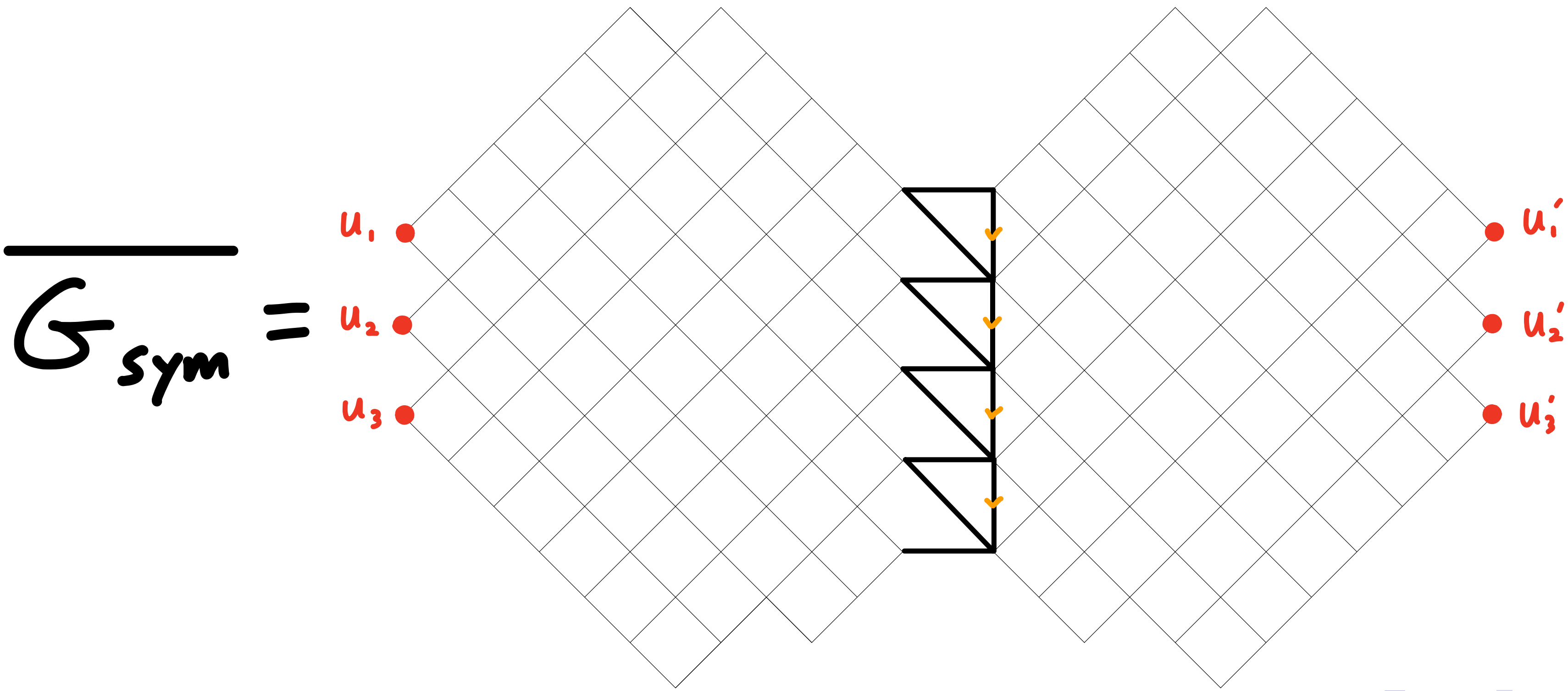
Construction of $\overline{G_{sym}}$

To construct $\overline{G_{sym}}$, from G_{sym}, we add

(1) n edges connecting v_i and v'_i for $i \in [n]$, directed toward v'_i,

(2) $(n-1)$ edges connecting v_i and v'_{i+1} for $i \in [n-1]$, directed toward v'_{i+1}, and

(3) $(n-1)$ edges connecting v'_i and v'_{i+1} for $i \in [n-1]$, directed toward v'_{i+1}



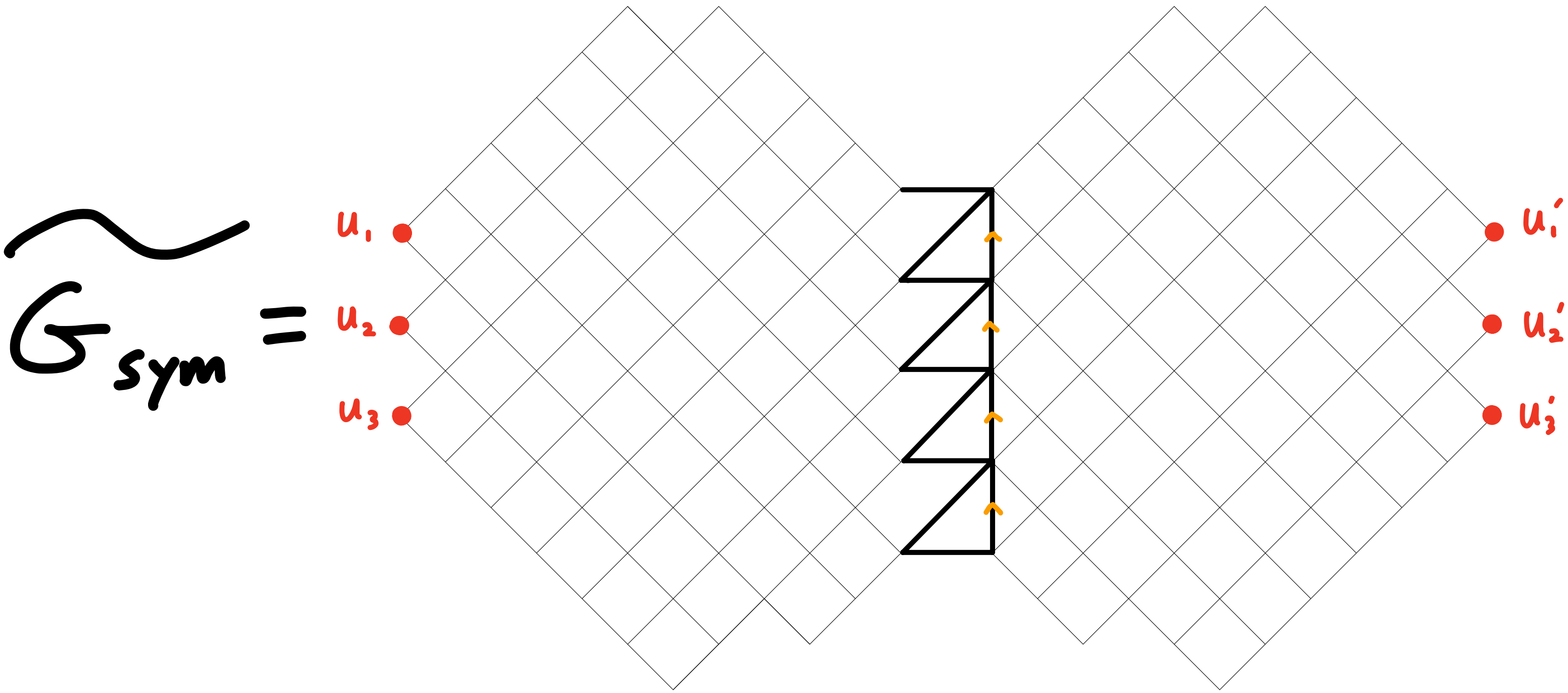
Construction of $\widetilde{G_{sym}}$

To construct $\widetilde{G_{sym}}$, from G_{sym}, we add

(1)′ n edges connecting v_i and v_i' for $i \in [n]$, directed toward v_i',

(2)′ $(n-1)$ edges connecting v_{i+1} and v_i' for $i \in [n-1]$, directed toward v_i', and

(3)′ $(n-1)$ edges connecting v_{i+1}' and v_i' for $i \in [n-1]$, directed toward v_i',



A reflection principle for nonintersecting paths

Let $\overline{\mathscr{P}_0}(\mathbf{u}, \mathbf{u}')$ be the sets of m-tuples of nonintersecting paths $(\overline{P_1}, \ldots, \overline{P_m})$ in $\overline{G_{sym}}$ such that $\overline{P_i} \in \mathscr{P}(u_i, u_i')$ for $i \in [m]$.

Similarly, let $\widetilde{\mathscr{P}_0}(\mathbf{u}, \mathbf{u}')$ be the sets of m-tuples of nonintersecting paths $(\widetilde{P_1}, \ldots, \widetilde{P_m})$ in $\widetilde{G_{sym}}$ such that $\widetilde{P}_i \in \mathscr{P}(u_i, u_i')$ for $i \in [m]$.

A reflection principle for nonintersecting paths

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A reflection principle for nonintersecting paths

If $\mathbf{u}$ and $\mathbf{v}$ are compatible, then

$$\mathrm{GF}[\mathscr{P}_0(\mathbf{u}, \mathbf{v})]^2 = \mathrm{GF}[\overline{\mathscr{P}_0}(\mathbf{u}, \mathbf{u}')] = \mathrm{GF}[\widetilde{\mathscr{P}_0}(\mathbf{u}, \mathbf{u}')].$$

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Similarly, let $\widetilde{\mathscr{P}_0}(\mathbf{u}, \mathbf{u}')$ be the sets of m-tuples of nonintersecting paths $(\widetilde{P_1}, \ldots, \widetilde{P_m})$ in $\widetilde{G_{sym}}$ such that $\widetilde{P_i} \in \mathscr{P}(u_i, u_i')$ for $i \in [m]$.

A reflection principle for nonintersecting paths

If $\mathbf{u}$ and $\mathbf{v}$ are compatible, then

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$\Rightarrow$ The enumeration of families of nonintersecting paths with unfixed ending points can be resolved by enumerating families of nonintersecting paths with fixed ending points instead!

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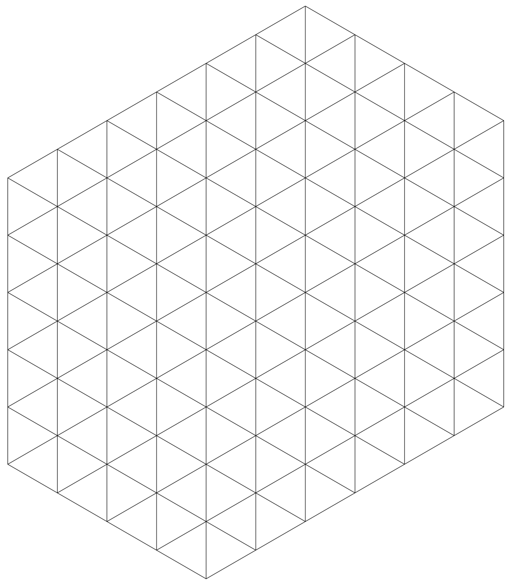
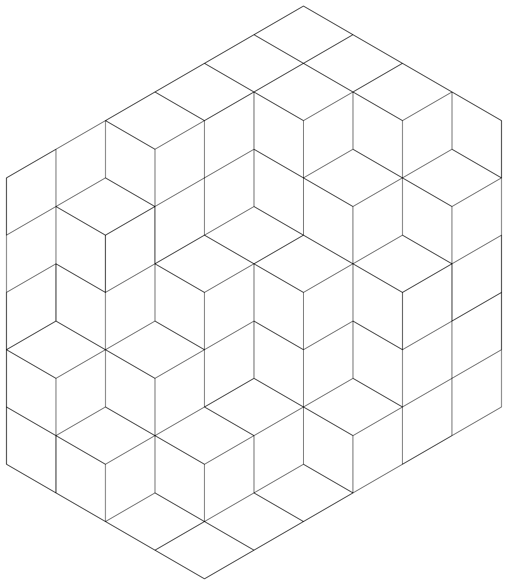
Lozenge tilings of regions on a triangular grid

Consider an arbitrary region R on a triangular grid, then one can ask the following question:





Q: How many lozenge tilings does R have?

Lozenge tilings of regions on a triangular grid

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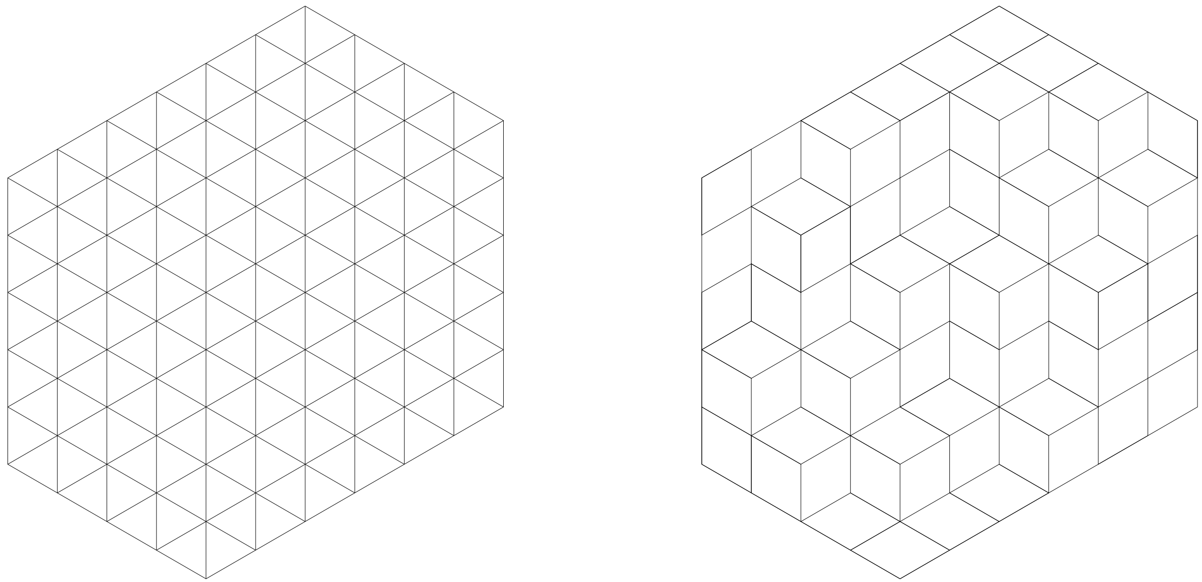
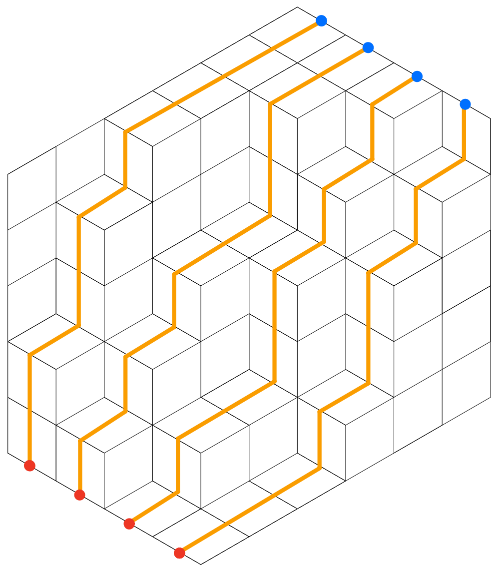
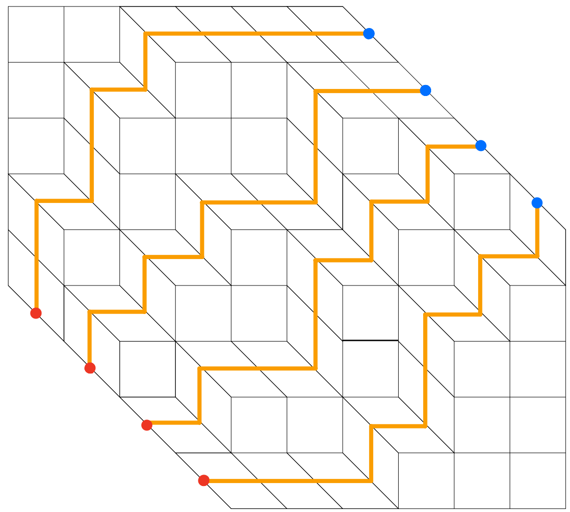


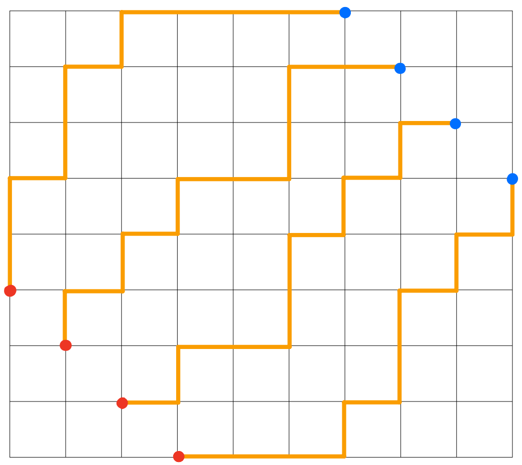
Q: How many lozenge tilings does R have?

It turns out that, in many cases, this question can be answered using nonintersecting paths enumeration techniques!

Lozenge tilings and nonintersecting paths on $\mathbb{Z}^2$

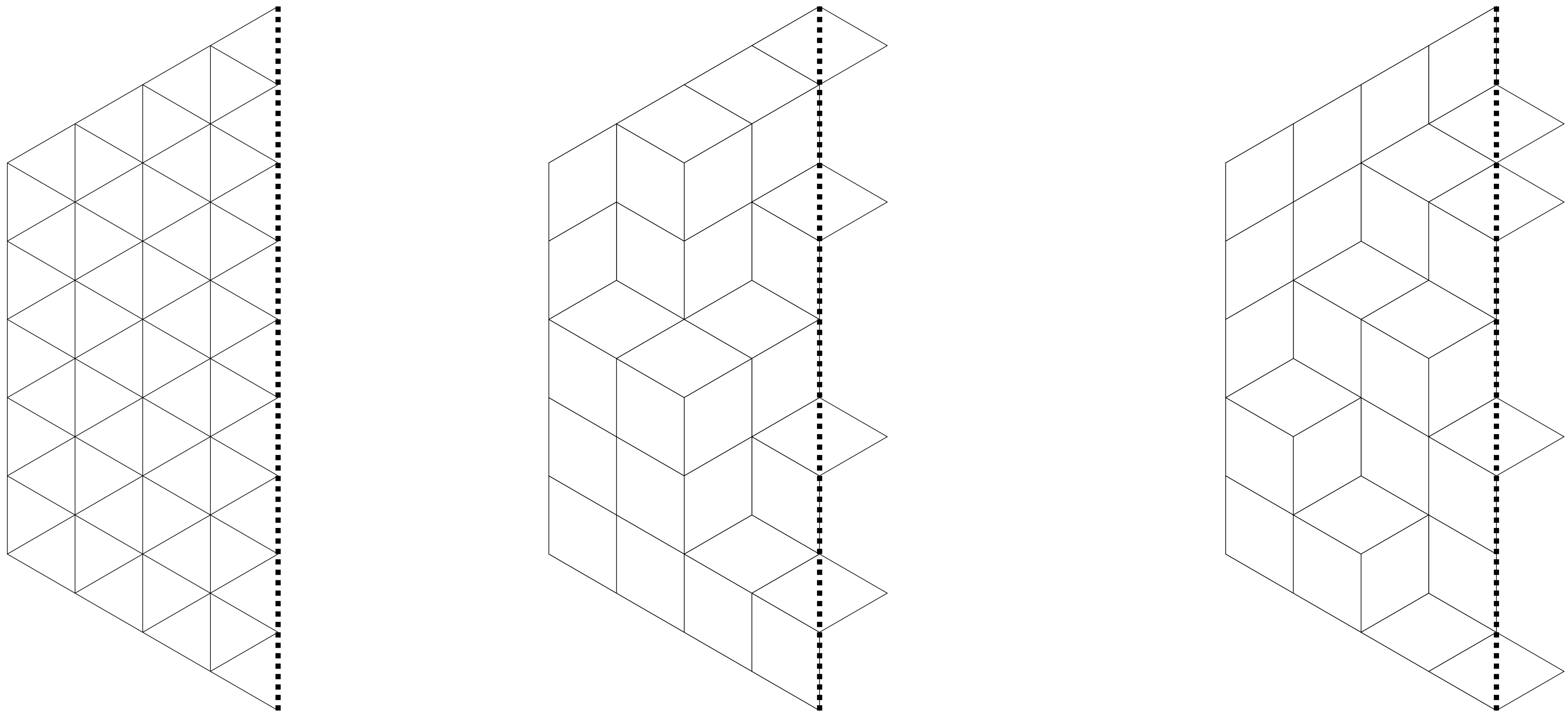




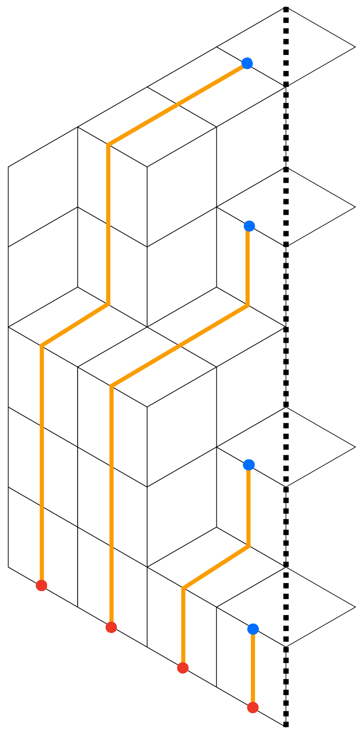


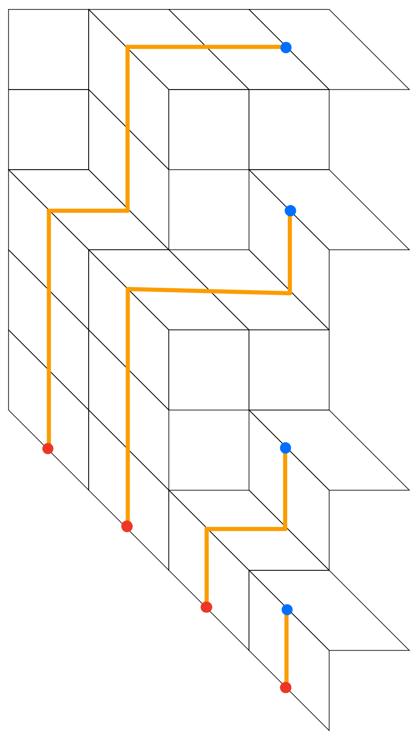
Lozenge tiling enumeration problems become more interesting if a region *R* has free boundaries!

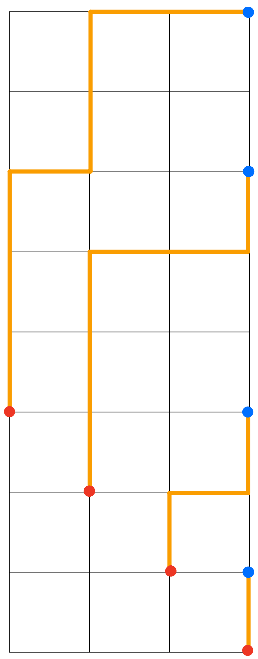
A boundary is *free* if lozenges are allowed to cross the boundary.



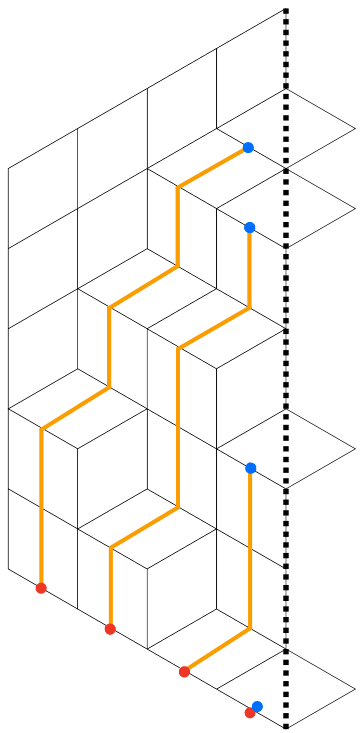
Free boundary lozenge tilings and nonintersecting paths.

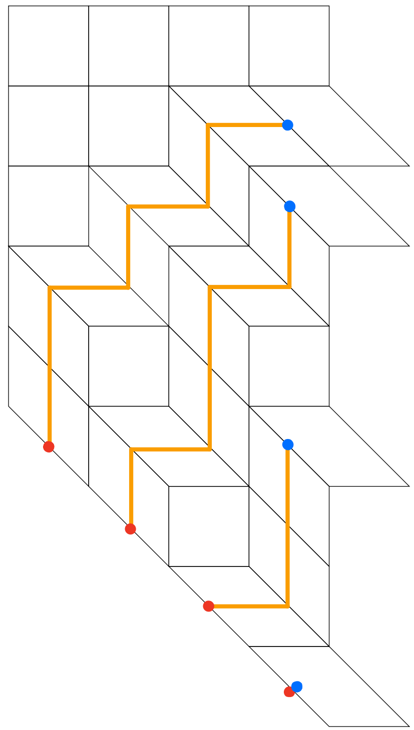


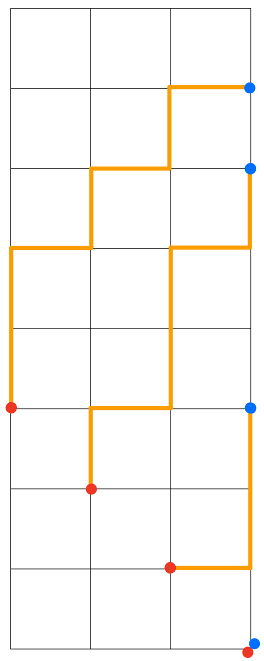




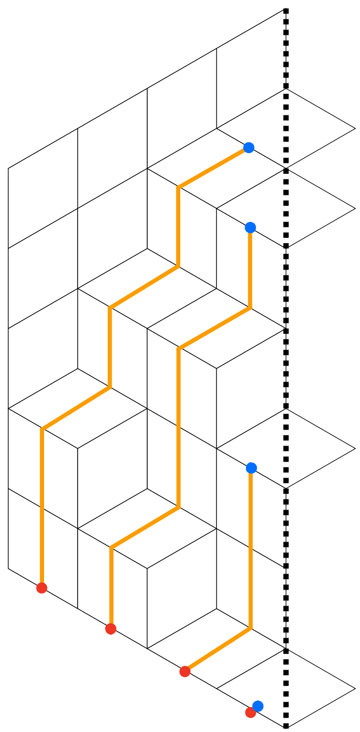
Free boundary lozenge tilings and nonintersecting paths.

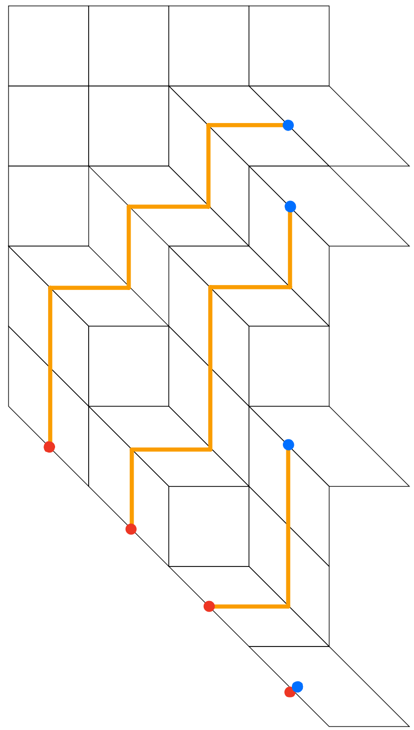


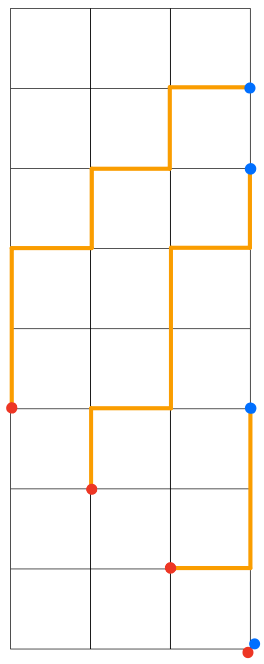




Free boundary lozenge tilings and nonintersecting paths.







Notice that the ending points of lattice paths are changed!

Lozenge tiling problems with/without free boundaries

Usually, lozenge tilings enumeration problems for regions with free boundaries are harder to solve, as one cannot apply Lindström–Gessel–Viennot theorem

When a region has a straight line free boundary, one can instead apply Okada–Stembridge's Pfaffian formula to enumerate the number of lozenge tilings

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As one application of our new formula, we can show that the enumeration of lozenge tilings of a large family of regions with free boundaries can be deduced from those without free boundaries.

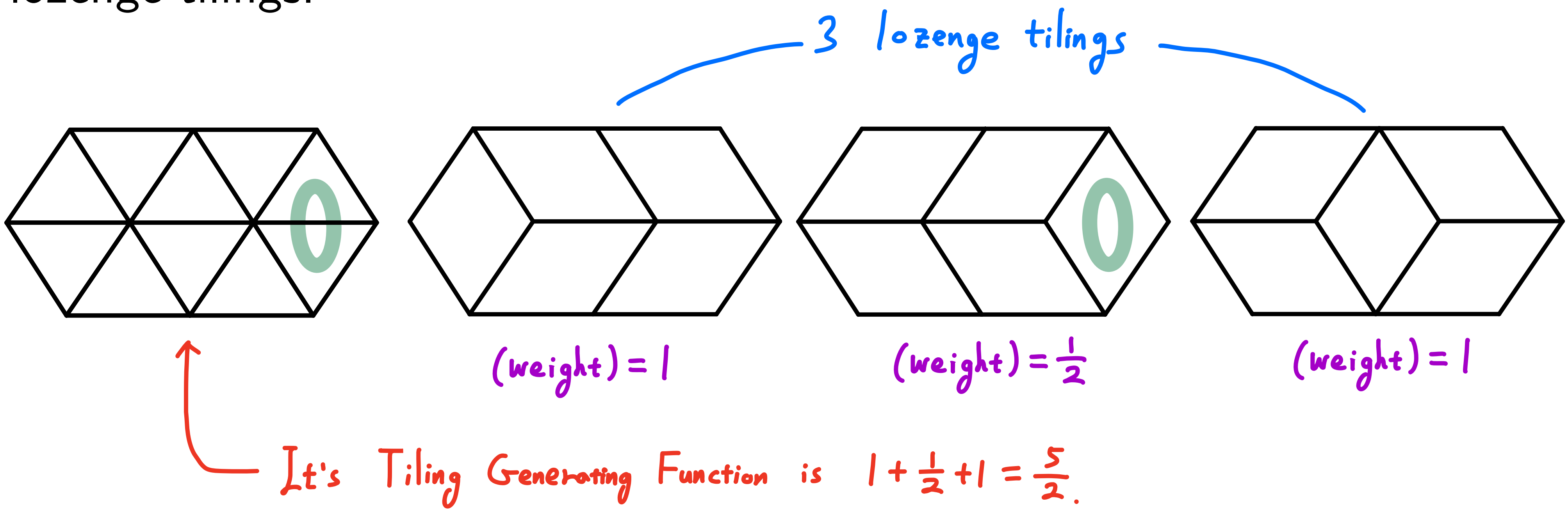
Lozenge tiling generating functions

In some cases, lozenges are weighted by some numbers or monomials.

In particular, in this talk, we will see some regions that have some lozenges weighted by $\frac{1}{2}$ (marked by shaded ellipses.)

A weight of a lozenge tiling is a product of weight of all lozenges that constitute the tiling.

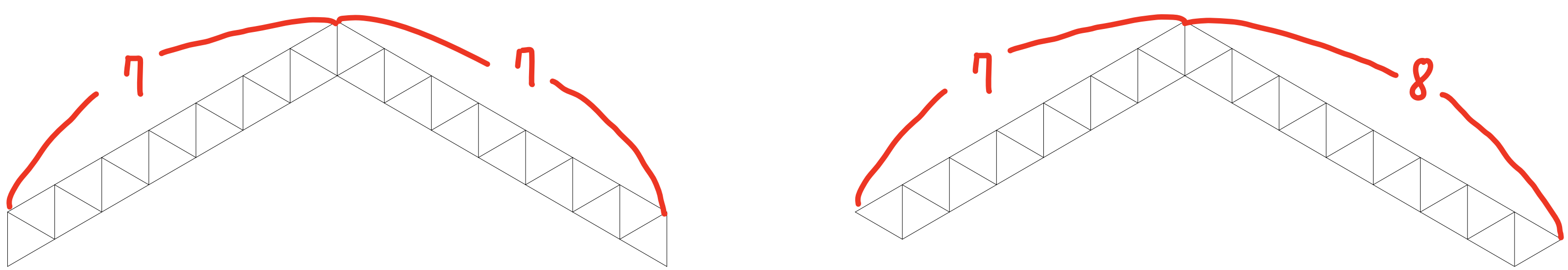
A tiling generating function of a region is the sum of weight of all its lozenge tilings.



A $\wedge$-hook and a shifted $\wedge$-hook

A $\wedge$-*hook of order n* is a $\wedge$-shaped hook that consists of $2n$ unit lozenges as described below.

A *shifted* $\wedge$*-hook of order n* is also a $\wedge$-shaped hook that is obtained from the $\wedge$-hook of order n by shifting the left-most unit triangle to the right end of it.



The $\wedge$-hook of order 7 (left) and the shifted $\wedge$-hook of order 7 (right).

Construction of two regions $A(m; \lambda_{st})$ and $\widetilde{A}(m; \lambda_{st})$

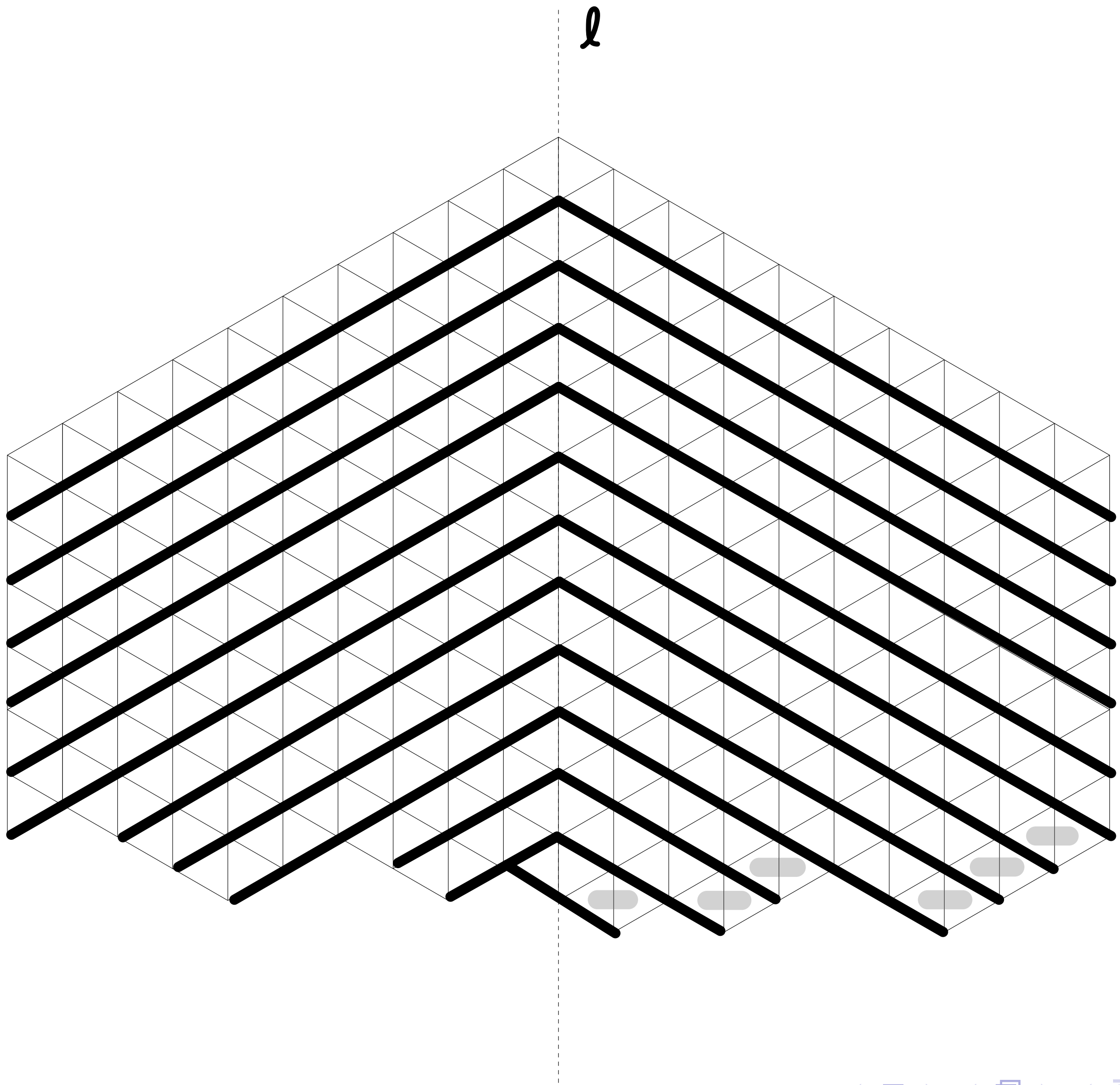
For any $m \in \mathbb{Z}_{\geq 0}$ and a strict partition $\lambda_{st} = (\lambda_1, \ldots, \lambda_k)$, consider m copies of $\wedge$-hook of order $(\lambda_1 + 1)$ and a copy of shifted $\wedge$-hook of order λ_i for $i \in [k]$.

(m copies) (k copies)

Then we concatenate these $\wedge$-hooks and shifted $\wedge$-hooks in order from top to bottom along the common axis ℓ, starting with the largest one in weakly decreasing order.

An example

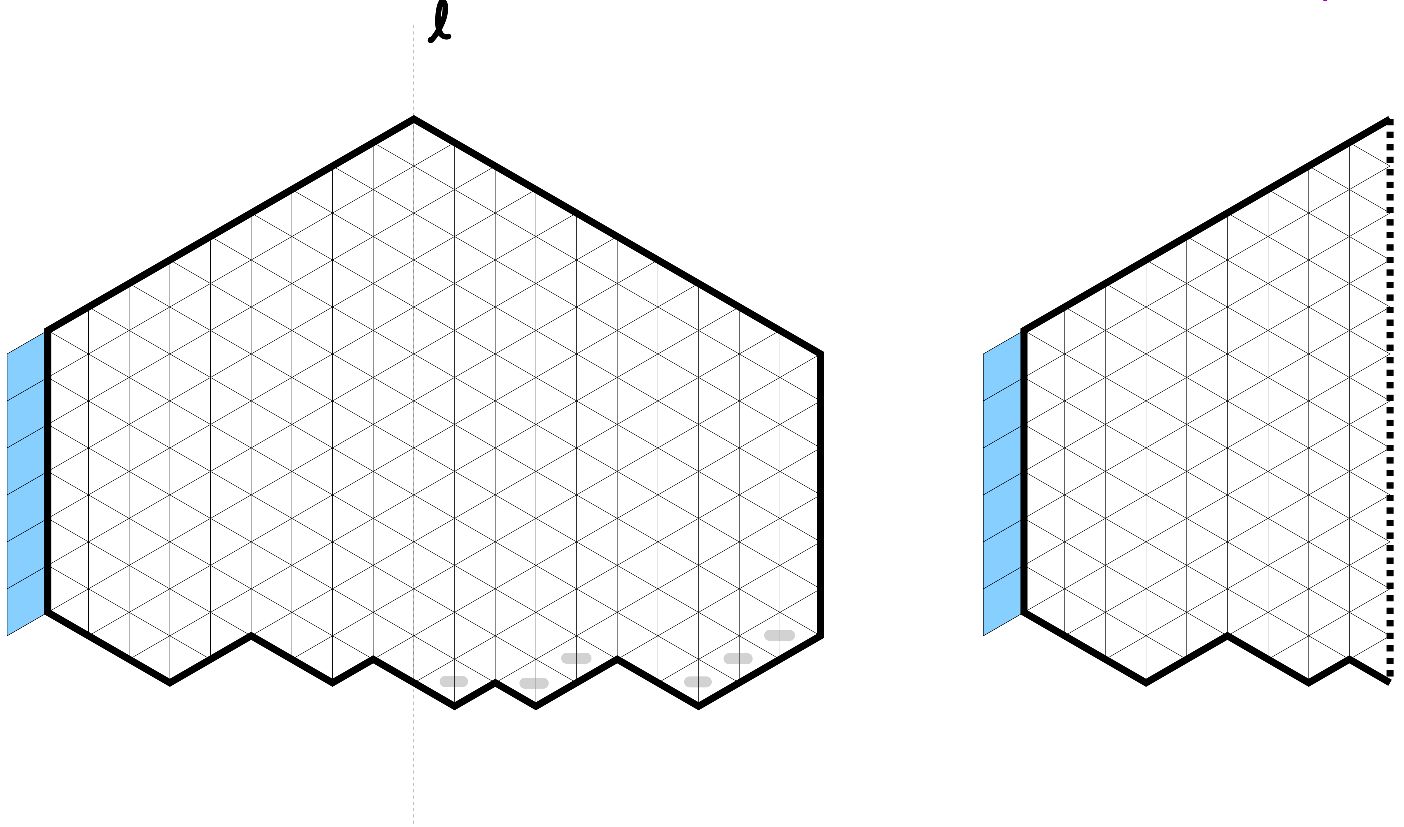
Example: $m = 6$ and $\lambda_{st} = (9, 8, 7, 4, 3, 1)$.



Two regions $A(m;\lambda_{st})$ and $\widetilde{A}(m;\lambda_{st})$

If we get rid of the leftmost strip from the previous regions, it is $\widetilde{A}(m;\lambda_{st})$. Every lozenge is weighted by 1, except the k horizontal lozenges at the right end of k shifted $\wedge$-hook of order λ_i for $i \in [k]$.

↳ they are weighted by $\frac{1}{2}$.



The subregion left to ℓ is $A(m;\lambda_{st})$ and its boundary along ℓ is free.

Two regions $A(m;\lambda_{st};I)$ and $\widetilde{A}(m;\lambda_{st};I)$

Let $I \subseteq [k]$ be a subset of $[k] = \{1,\ldots,k\}$.
(recall that k is the number of parts of λ_{st}.)

We mark the leftmost and rightmost unit triangle in the shifted $\wedge$-hook of order λ_i by i and i', respectively.

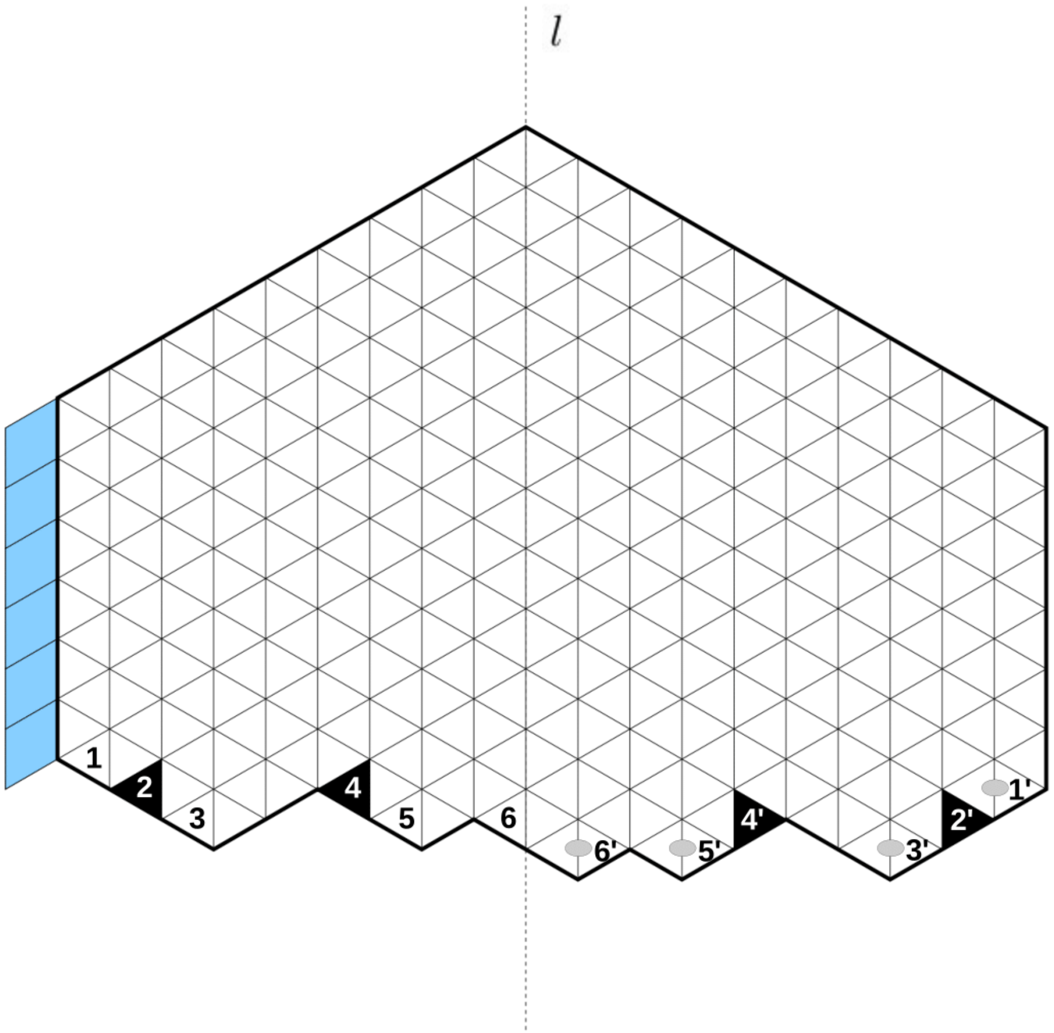
Then $\widetilde{A}(m;\lambda_{st};I)$ is obtained from $\widetilde{A}(m;\lambda_{st})$ by deleting unit triangles labeled by i and i' for $i \in I$.

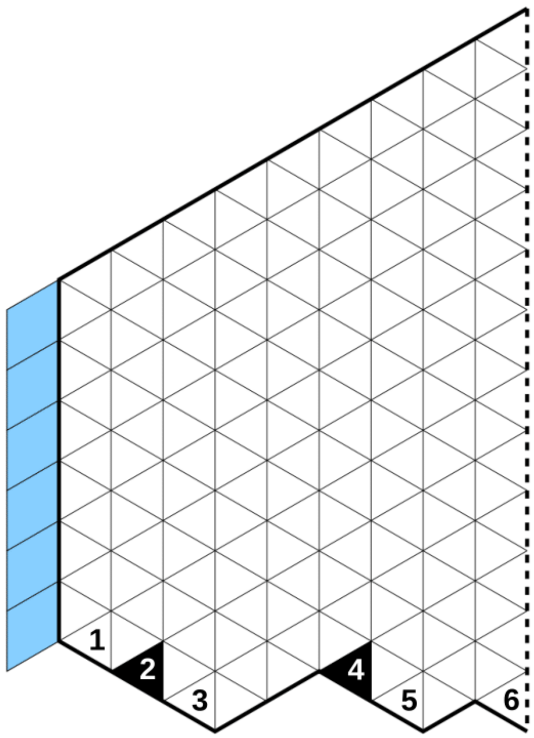
Similarly, $A(m;\lambda_{st};I)$ is obtained from $A(m;\lambda_{st})$ by deleting unit triangles labeled by the elements of I

Note: $A(m;\lambda_{st})$ and $\widetilde{A}(m;\lambda_{st})$ are special cases of $A(m;\lambda_{st};I)$ and $\widetilde{A}(m;\lambda_{st};I)$ when $I = \varnothing$, respectively.

Examples of $A(m; \lambda_{st}; I)$ and $\widetilde{A}(m; \lambda_{st}; I)$

$m = 6$, $\lambda_{st} = (9, 8, 7, 4, 3, 1)$, and $I = \{2, 4\}$.





A new theorem

Let $\mathrm{M}_f(A(m; \lambda_{st}; I))$ be the number of lozenge tilings of $A(m; \lambda_{st}; I)$.

Let $\mathrm{M}(\widetilde{A}(m; \lambda_{st}; I))$ be the tiling generating function of $\widetilde{A}(m; \lambda_{st}; I)$.

Theorem (Byun, 2025)

For a nonnegative integer m, a strict partition λ_{st} with k parts, and a set $I \subseteq [k]$,

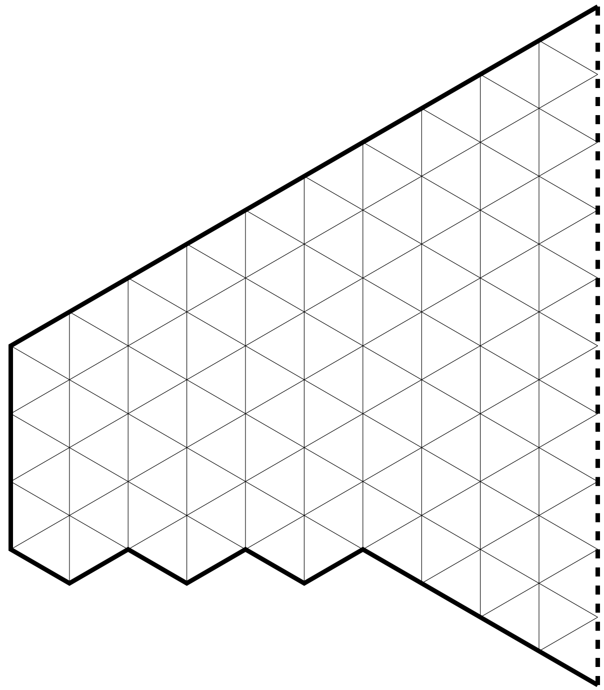
$$\mathrm{M}_f(A(m; \lambda_{st}; I))^2 = 2^{k-|I|}\,\mathrm{M}(\widetilde{A}(m; \lambda_{st}; I)).$$

Proof idea: The proof uses our earlier determinant formula combined with properties of the matrix U.

Special cases of $A(m; \lambda_{st}; I)$ and $\widetilde{A}(m; \lambda_{st}; I)$ (1/3)

Special cases of the regions $A(m; \lambda_{st}; I)$ and $\widetilde{A}(m; \lambda_{st}; I)$ appeared in several literature.

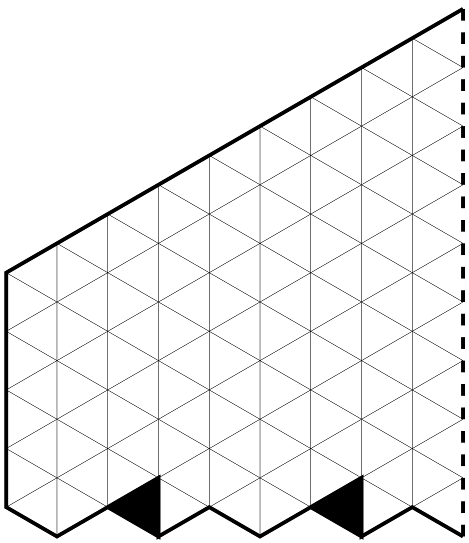
When $I = \varnothing$ and $\lambda_{st} = (n, \ldots, 1) + (k, \ldots, 1)$ for $0 \leq k \leq n$:

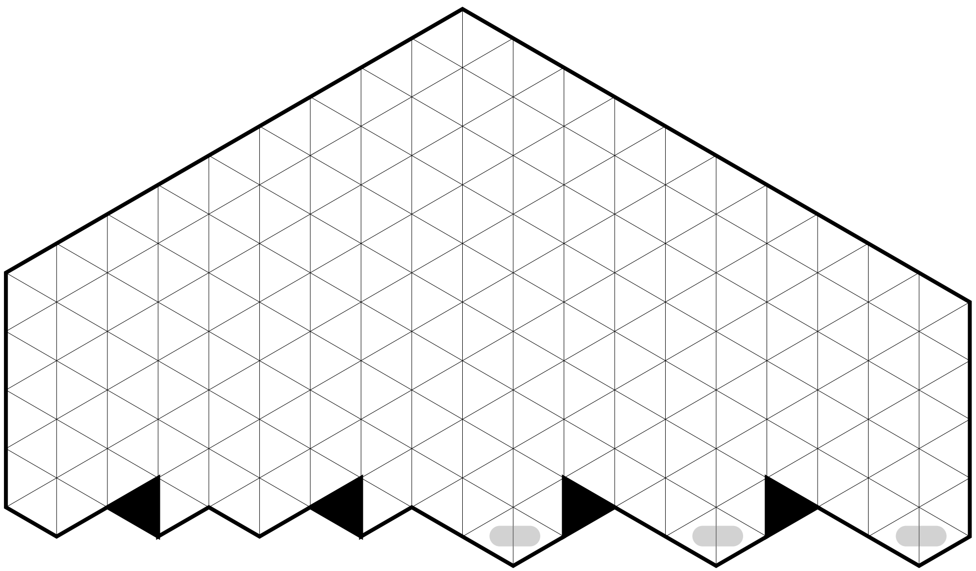


This region is related to "Plane partitions of shifted double staircase shape" (studied independently by Hopkins–Lai and Okada.)

Special cases of $A(m; \lambda_{st}; I)$ and $\widetilde{A}(m; \lambda_{st}; I)$ (2/3)

When $I \subseteq [k]$ is arbitrary and $\lambda_{st} = (n-1, n-3, \ldots, 1 \text{ or } 2)$:





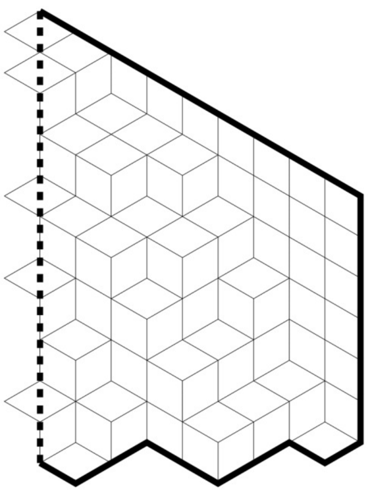
These regions are related to "Symmetry classes of lozenge tilings of Punctured hexagons" (both regions are studied by Ciucu.)
If $I = \varnothing$, they are related to "Symmetry classes of boxed plane partitions" (studied by Proctor and Stanley, respectively.)

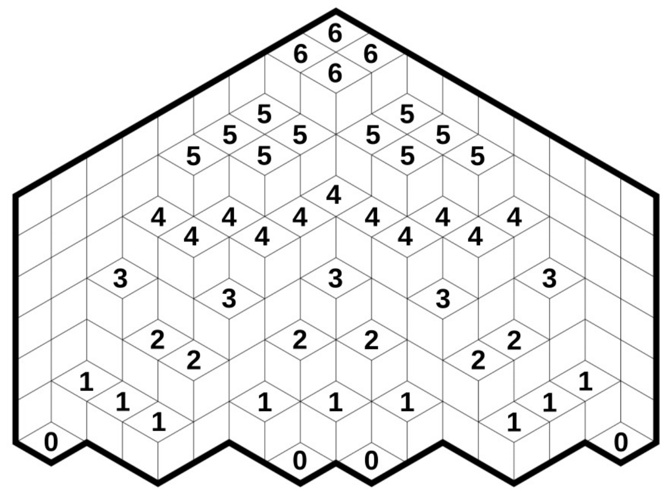
↑ SSCPP ↑ SCPP

When $I = \varnothing$ and λ_{st} is arbitrary:

6	6	5	5	5	4	3	1	0
	6	5	5	4	4	2	1	
		4	4	4	3	2	1	
			3	2	1			
				1	0			

6	6	5	5	5	4	3	1	0
6	6	5	5	4	4	2	1	
5	5	4	4	4	3	2	1	
5	5	4	3	2	1			
5	4	4	2	1	0			
4	4	3	1	0				
3	2	2						
1	1	1						
0								





This region is related to "Shifted plane partitions of an arbitrary shifted shape and Symmetric plane partitions of an arbitrary symmetric shape."

Thank you!