About Permutation Matrices

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Summary

The study of **permutations** is both ancient and modern. They can be viewed as the integers $1, 2, \ldots, n$ in some order or as $n \times n$ permutation matrices. They can be regarded as data which is to be sorted. The explicit definition of the determinant uses permutations. An **inversion** of a permutation occurs when a larger integer precedes a smaller integer. Inversions can be used to define two **partial orders** on permutations, one weaker than the other. Partial orders have a unique minimal completion to a **lattice**, the **Dedekind-MacNeille completion**. Generalizations of permutation matrices determine related matrix classes, for instance, **alternating sign matrices (ASMs)** which arose independently in the mathematics and physics literature. Permutations may contain certain **patterns**, e.g. three integers in increasing order; avoiding such patterns determines certain permutation classes. Similar restrictions can be placed more generally on $(0, 1)$-matrices. The convex hull of $n \times n$ permutation matrices is the **polytope** of $n \times n$ doubly stochastic matrices. In a similar way we get **ASM polytopes**. We shall explore these and other ideas and their connections.

A Permutation Primer (I)

Permutations can be modeled in two ways:

- As a listing of a set of n elements, usually take to be the integers $\{1, 2, \ldots, n\}$, in some order, e.g. if $n = 6$, $(3, 6, 1, 5, 2, 4)$.
- As a (permutation) matrix, e.g.

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \text{ or as } \begin{bmatrix} & & 1 & & & \\ & & & & & 1 \\ 1 & & & & & \\ & & & & 1 & \\ & 1 & & & & \\ & & & 1 & & \end{bmatrix}.$$

MathSciNet: $> 31,000$ hits

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A Permutation Primer (II)

So a permutation consists of the integers $\{1, 2, \ldots, n\}$ in some order, and as a result some of the integers are **out of order**. How to measure this? **Number of inversions.**

Let $\sigma = (k_1, k_2, \ldots, k_n)$ be a permutation of $\{1, 2, \ldots, n\}$.

- (k_p, k_q) is an **inversion** of σ provided

  $$p < q \text{ and } k_p > k_q \text{ (a pair of integers out of their natural order).}$$

  The transformation $(k_p, k_q) \to (k_q, k_p)$ is a **transposition**. Returning to our example, $(3, \mathbf{6}, 1, 5, \mathbf{2}, 4)$.

		1			
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A Permutation Primer (III)

A transposition can always be chosen to reduce the **number**, not necessarily the **set of inversions** $\mathcal{I}(\sigma)$, by 1:

$$(\mathbf{3}, 4, 1, \mathbf{2}) \to (\mathbf{2}, 4, 1, \mathbf{3})$$

reduces the **number** of inversions from 4 to 3 but not the set of inversions by 1:

$$\{(3,1),(3,2),(4,1),(4,2)\} \to \{(2,1),(4,1),(4,3)\}.$$

Adjacent inversion is of the form (k_p, k_{p+1}) with $k_p > k_{p+1}$. Effect of the corresponding transposition $(k_p, k_{p+1}) \to (k_{p+1}, k_p)$ is to remove one inversion from $\mathcal{I}(\sigma)$.

$$(3, \mathbf{4}, \mathbf{1}, 2) \to (3, \mathbf{1}, \mathbf{4}, 2) :$$

If there is an inversion (so not the identity $(1, 2, \ldots, n)$), then there must be an adjacent inversion.

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Basic Fact:

- A permutation σ of $\{1, 2, \ldots, n\}$ is uniquely determined by its set $\mathcal{I}(\sigma)$ of inversions (use induction on the location of 1) but not, in general, by its set of adjacent inversions. For example,
- Permutations $(4, 1, 2, 3)$ and $(2, 4, 1, 3)$ have exactly one adjacent inversion, namely $(4, 1)$ in both instances, but their sets of inversions are different: $\{(4, 1), (4, 2), (4, 3)\}$ and $\{(2, 1), (4, 1), (4, 3)\}$, respectively.

How can we compare two permutations, other than by using the number of inversions? By a partial order, in fact, two partial orders.

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A Permutation Primer (V): Two Partial Orders

- **Weak Bruhat Order** on permutations of $\{1, 2, \ldots, n\}$:

$$\pi_1 \preceq_b \pi_2 \text{ provided that } \mathcal{I}(\pi_1) \subseteq \mathcal{I}(\pi_2).$$

  Equivalent to: π_1 can be obtained from π_2 by a sequence of **adjacent transpositions**, each thereby reducing the **set** of inversions by exactly 1).

- **Bruhat order**: $\pi_1 \preceq_B \pi_2$ provided π_1 can be obtained from π_2 by a sequence of **transpositions** each reducing the **number** of inversions by exactly 1, but not necessarily reducing the set of inversions by 1.
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- **Set Containment** versus **Number**.

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Example

- $(4, 2, 1, 3) \preceq_B (4, 3, 1, 2)$ since

$$(4, \mathbf{2}, 1, \mathbf{3}) \preceq_B (4, \mathbf{3}, 1, \mathbf{2}) \text{ (one transposition)}$$

- $(4, 2, 1, 3) \not\preceq_b (4, 3, 1, 2)$, since $\mathcal{I}((4, 2, 1, 3)) = \{(4, 2), (4, 1), (4, 3), (2, 1)\} \not\subseteq \mathcal{I}((4, 3, 1, 2)) = \{(4, 3), (4, 1), (4, 2), (3, 1), (3, 2)\}$.

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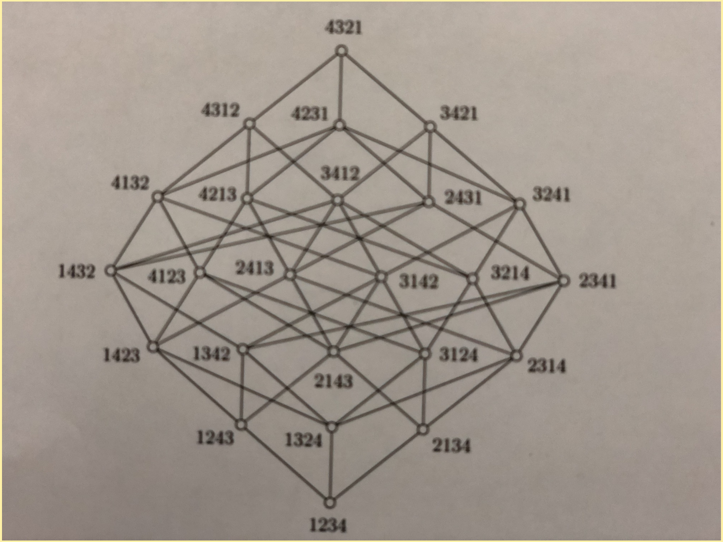
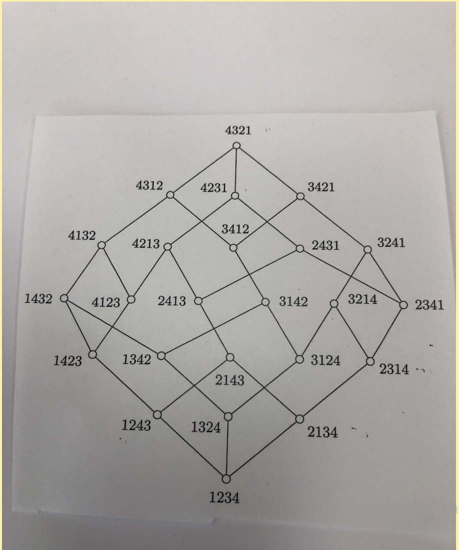
Bruhat order on $\mathcal{S}_4$ (Bjőrner& Brenti book): $(4,2,1,3) \preceq_B (4,3,1,2)$



Weak Bruhat order on $\mathcal{S}_4$: (Bjőrner& Brenti book): $(4,2,1,3) \not\leq_b (4,3,1,2)$



Weak Bruhat order

A **lattice**: **any two elements have an LUB and a GLB** (for finite partially ordered sets LUBs (resp. GLBs) guarantee GLBs (resp. LUBs).

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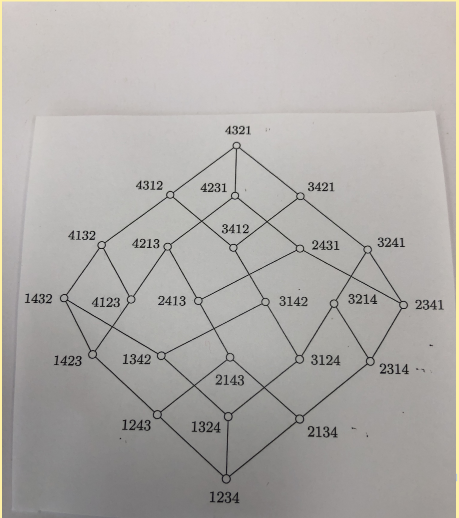

Bruhat order

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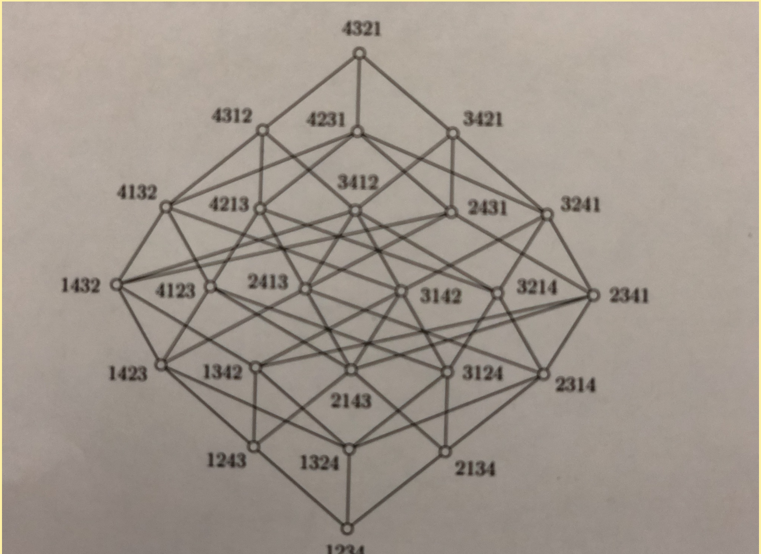
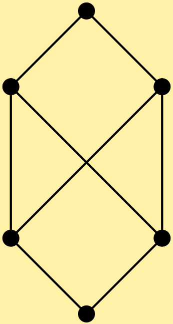
Not a lattice e.g. GLB(4312,4231) not defined:



Bruhat order on the permutations of order 3: $(\mathfrak{S}_3, \preceq_B)$

$$L_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} : (3, 2, 1)$$

$$(2, 3, 1) : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} : (3, 1, 2)$$



$$(2, 1, 3) : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} : (1, 3, 2)$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : (1, 2, 3)$$

Bruhat Orders on Permutations

In both orders, the **identity** $\iota_n = (1, 2, \ldots, n)$ (I_n) is the unique minimal element $(\mathcal{I}(\iota_n) = \emptyset)$, and the **anti-identity** $\zeta_n = (n, n-1, \ldots, 2, 1)$ is the unique maximal element $(\mathcal{I}(\zeta_n) = \{(i,j) : i > j\})$.

The **cover relation** is given by:

- $\pi_1 \preceq_b \pi_2$, and $\mathcal{I}(\pi_1)$ is obtained from $\mathcal{I}(\pi_1)$ by removing one inversion.
- $\pi_1 \preceq_B \pi_2$, and π_1 has exactly 1 fewer inversion.

Example: $(4, 2, 1, 5, 3) \preceq_B (4, 5, 1, 2, 3)$ Both Bruhat orders on $\mathfrak{S}_n$ are

graded by the number of inversions. The **grade** corresponds to the **level** in the diagram of the partially ordered set.

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graded by the number of inversions. The **grade** corresponds to the **level** in the diagram of the partially ordered set.

Bruhat Orders on Permutations

In both orders, the **identity** $\iota_n = (1, 2, \ldots, n)$ (I_n) is the unique minimal element $(\mathcal{I}(\iota_n) = \emptyset)$, and the **anti-identity** $\zeta_n = (n, n-1, \ldots, 2, 1)$ is the unique maximal element $(\mathcal{I}(\zeta_n) = \{(i, j) : i > j\})$.

The **cover relation** is given by:

- $\pi_1 \preceq_b \pi_2$, and $\mathcal{I}(\pi_1)$ is obtained from $\mathcal{I}(\pi_1)$ by removing one inversion.
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Bruhat Order: The Σ-way

For an $m \times n$ matrix $A = [a_{ij}]$, define $\Sigma(A) = [\sigma_{ij}(A)]$ by

$$\sigma_{ij} = \sigma_{ij}(A) = \sum_{1 \le k \le i, 1 \le l \le j} a_{ij}, \quad (1 \le i \le m, 1 \le j \le n)$$

the sum of the entries of the leading $i \times j$ submatrix of A. (If A is a permutation matrix, this is the same as the rank of the leading $i \times j$ submatrix of A.)

Example: $A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 3 & 1 & 2 \\ 3 & 5 & 1 & 2 \end{bmatrix} \to \Sigma(A) = \begin{bmatrix} 1 & 4 & 6 & 10 \\ 1 & 7 & 10 & 16 \\ 4 & 15 & 19 & 27 \end{bmatrix}$

Theorem: For $n \times n$ permutation matrices P and Q, we have

$$P \preceq_B Q \text{ if and only if } \Sigma(P) \ge \Sigma(Q) \text{ (entrywise).}$$

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Dedekind-MacNeille Completion of a Partially Ordered Set

Theorem (MacNeille 1937): Let $(P, \leq_P)$ be a finite partially ordered set. Then there exists a unique **minimal** lattice $(L, \leq_L)$ such that $P \subseteq L$ and for $a, b \in P$, $a \leq_P b$ if and only if $a \leq_L b$. $(L, \leq_L)$ is the **Dedekind-MacNeille completion** of $(P, \leq_P)$.

So if you have a favorite partially ordered set which is not a lattice, you can try to find its Dedekind-MacNeille completion. The Dedekind-MacNeille completion of the rational numbers with the usual order gives the real numbers with $\pm\infty$.

Recall the Bruhat order on the permutations of order 3: $(\mathfrak{S}_3, \preceq_B)$, repeated on the next slide.

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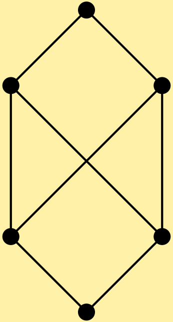
Bruhat order on the permutations of order 3: $(\mathfrak{S}_3, \preceq_B)$

(**Not a lattice**)

$$L_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} : (3,2,1)$$

$$(2,3,1) : \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} : (3,1,2)$$



$$(2,1,3) : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} : (1,3,2)$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : (1,2,3)$$

What is the Dedekind-MacNeille Completion of $(\mathfrak{S}_n \preceq_B)$?

What are the new elements? Let's do it for $n = 3$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{\Sigma_1} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\Sigma_2} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

do not have a meet: With $\Sigma_3 = \min\{\Sigma_1, \Sigma_2\} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, there does not exist a permutation matrix with this Σ_3. The problem is the 1 in the $(2, 2)$-position of Σ_3. But

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\Sigma_3} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

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Dedekind-MacNeille Completion of $(\mathfrak{S}_3, \preceq_B)$

$$L_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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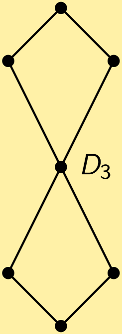

$$D_3 \text{ where } D_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

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Dedekind-MacNeille Completion of $(\mathfrak{S}_n, \preceq_B)$, Version # 1

Theorem (Lascoux & Schűtzenberger 1996): The Dedekind-MacNeille completion of $(\mathfrak{S}_n, \preceq_B)$ is:

$$\Sigma_n = \{X = [x_{ij}], n \times n \text{ nonnegative integral matrix } \}$$

such that

- For each i, the integers in row i and column i are taken from $\{1, 2, \ldots, i\}$ beginning with 0 or 1 and ending with i,
- For each i, the integers in row i and column i are nondecreasing.
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In particular, the last row and last column contain $1, 2, \ldots, n$ in that order.

The (lattice) partial order is the entrywise order.

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Dedekind-MacNeille Completion of $(\mathfrak{S}_n, \preceq_B)$: Version # 2

Theorem (Lascoux & Schützenberger 1996): The MacNeille completion of $(\mathfrak{S}_n, \preceq_B)$ is $(\mathfrak{A}_n, \preceq_B)$ where

- $\mathfrak{A}_n$ is the set of $n \times n$ **alternating sign matrices**: $(0, 1, -1)$-matrices where the ± 1's in each row and column alternate, ignoring 0's, and start and end with a 1.
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Note that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, and thus these transformations are transpositions where -1s are now allowed in the result. All $n \times n$ ASMs can be obtained from I_n by a sequence of transpositions with all intermediary matrices ASMs.

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Examples of non-Permutation ASMs

$$
\begin{bmatrix}
 & & 1 & & & \\
 & 1 & -1 & & 1 & \\
1 & -1 & & 1 & -1 & 1 \\
 & & 1 & -1 & 1 & \\
 & 1 & -1 & 1 & & \\
 & & 1 & & &
\end{bmatrix},
\quad
\begin{bmatrix}
 & 1 & & & & \\
1 & -1 & 1 & & & \\
 & 1 & -1 & 1 & & \\
 & & 1 & -1 & 1 & \\
 & & & 1 & -1 & 1 \\
 & & & & 1 &
\end{bmatrix}.
$$

Some Basic Properties of ASMs

- The partial row and column sums starting from the first or last entry equal 0 or 1, with the full row and column sums equal to 1.
- The ASM property is preserved under the dihedral group of order 8 (symmetries of a square), but not under arbitrary (simultaneous) row and columns permutations.
- The 6×6 diamond ASMs (largest number of nonzeros) where we use $\pm$ in place of ± 1:

$$D_6 = \left[\begin{array}{c|c|c|c|c|c} & & + & & & \\ \hline & + & - & + & & \\ \hline + & - & + & - & + & \\ \hline & + & - & + & - & + \\ \hline & & + & - & + & \\ \hline & & & + & & \end{array}\right], \quad D_6' = \left[\begin{array}{c|c|c|c|c|c} & & & + & & \\ \hline & & + & - & + & \\ \hline & + & - & + & - & + \\ \hline + & - & + & - & + & \\ \hline & + & - & + & & \\ \hline & & + & & & \end{array}\right]$$

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Bijection between the Two Versions of the Dedekind-MacNeille Completion

- If A is an $n \times n$ ASM, Then $\Sigma(A)$ satisfies the conditions of Σ_n:
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- Given a matrix $X = [x_{ij}] \in \Sigma_n$, then $A = [a_{ij}]$ is an $n \times n$ ASM where

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Enumeration of ASMs

The number of $n \times n$ permutation matrices is $n!$. **How many $n \times n$ ASMs are there?**

- For small n, the number of $n \times n$ ASMs is: $1, 2, 7, 42, 429, 7436, \ldots$.
- Mills, Robbins, and Rumsey made a conjecture in 1983.
- **Celebrated Theorem of Zeilberger 1996, and later and independently by Kuperberg**: The number of $n \times n$ ASMs is

$$\frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} = \prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!} \sim \left(\frac{3\sqrt{3}}{4}\right)^{n^2}.$$

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ASMs and Square Ice

There is a 1-1 correspondence between ASMs and something called "square ice" configurations: a system of water (H_2O) molecules frozen in a square lattice.

Square Ice I

There are oxygen atoms at each vertex of an $n \times n$ lattice, with hydrogen atoms between successive oxygen atoms in a row or column, and on either vertical side of the lattice, but not on the two horizontal sides. E.G. $n = 4$:

$$\begin{array}{ccccccccc} H & O & H & O & H & O & H & O & H \\ & H & & H & & H & & H & \\ H & O & H & O & H & O & H & O & H \\ & H & & H & & H & & H & \\ H & O & H & O & H & O & H & O & H \\ & H & & H & & H & & H & \\ H & O & H & O & H & O & H & O & H \end{array}.$$

Each O is to be attached to two Hs (a water molecule H_2O) in a one to two bijection. There are six possible configurations in which an oxygen atom can be attached to two hydrogen atoms:

Square Ice II

$$
H \leftarrow O \rightarrow H \qquad \begin{array}{c} H \\ \uparrow \\ O \\ \downarrow \\ H \end{array}
$$

$$
\begin{array}{ccc} & & H \\ & & \uparrow \\ H & \leftarrow & O \end{array} \qquad
\begin{array}{ccc} H & & \\ \uparrow & & \\ O & \rightarrow & H \end{array} \qquad
\begin{array}{ccc} O & \rightarrow & H \\ \downarrow & & \\ H & & \end{array} \qquad
\begin{array}{ccc} H & \leftarrow & O \\ & & \downarrow \\ & & H \end{array} .
$$

Let the top left (**horizontal**) configuration correspond to $\mathbf{1}$ and the top right (**vertical**) configuration correspond to $-\mathbf{1}$. Let the other four (**skew**) configurations correspond to $\mathbf{0}$.

Square Ice III

$(n = 4)$

$$
\begin{array}{ccccccccccccccccc}
H & \leftarrow & O & & H & \leftarrow & O & & H & \leftarrow & O & \rightarrow & H & & O & \rightarrow & H \\
 & & \downarrow & & & & \downarrow & & & & & & & & \downarrow & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & & & & & & & & & \uparrow & & & & & & \\
H & \leftarrow & O & \rightarrow & H & & O & \rightarrow & H & & O & & H & \leftarrow & O & \rightarrow & H \\
 & & & & & & \downarrow & & & & \downarrow & & & & & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & \uparrow & & & & & & & & & & & & \uparrow & & \\
H & \leftarrow & O & & H & \leftarrow & O & \rightarrow & H & & O & \rightarrow & H & & O & \rightarrow & H \\
 & & & & & & & & & & \downarrow & & & & & & \\
 & & H & & & & H & & & & H & & & & H & & \\
 & & \uparrow & & & & \uparrow & & & & & & & & \uparrow & & \\
H & \leftarrow & O & & H & \leftarrow & O & & H & \leftarrow & O & \rightarrow & H & & O & \rightarrow & H
\end{array}
$$

and this corresponds to the ASM: $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

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Origin of ASMs: the λ-determinant

The λ-determinant arises by starting with

$$\det_\lambda \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] = a_{11}a_{22} + \lambda a_{12}a_{21} \text{ (or with } \det_\lambda[a_{11}] = a_{11}\text{)}$$

and adapting the well-known Dodgson's condensation formula for determinants (which iteratively expresses a determinant in terms of 2×2 determinants) to the λ-determinant using the rule

$$\det_\lambda A = \frac{\det_\lambda A_{UL} \det_\lambda A_{LR} + \lambda \det_\lambda A_{UR} \det_\lambda A_{LL}}{\det_\lambda A_C}.$$

(A_{UL} is the $(n-1) \times (n-1)$ submatrix in upper left, A_{LR} in lower right, etc. and A_C is the $(n-2) \times (n-2)$ submatrix in the center.)

If $\lambda = -1$, we get Dodgson's formula for the ordinary determinant.

The λ-determinant

If $n = 2$ (so C is empty), we get

$$\det_\lambda \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} + \lambda a_{12}a_{21}.$$

(if $\lambda = -1$, we get the ordinary determinant)

If $n = 3$ (so $C = [a_{22}]$) we get

$$\det_\lambda(A) = a_{11}a_{22}a_{33} + \lambda a_{12}a_{21}a_{33} + \lambda a_{11}a_{23}a_{32} + (\lambda^2 + \lambda)a_{12}a_{21}a_{22}^{-1}a_{23}a_{32}$$

$$+\lambda^2 a_{13}a_{21}a_{32} + \lambda^2 a_{12}a_{23}a_{31} + \lambda^3 a_{13}a_{22}a_{31}.$$

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The λ-determinant and ASMs

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If for each of the seven terms we replace entries in A by the corresponding power we get the seven 3×3 ASMs. For instance,

$$(\lambda^2+\lambda)a_{12}a_{21}a_{22}^{-1}a_{23}a_{32} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and the other terms give the six 3×3 permutation matrices.

If $A=[a_{ij}]$ is an $n\times n$ matrix, then $\det_\lambda A$ is of the form

$$\sum_{B=[b_{ij}]\in \mathrm{ASM}_{n\times n}} p_B(\lambda)\prod_{i,j=1}^{n} a_{ij}^{b_{ij}}$$

where $p_B(\lambda)$ is a polynomial in λ. **The number of terms is $|\mathrm{ASM}_{n\times n}|$.**

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Avoiding Patterns in Permutations I

Permutation Patterns form a huge topic (there is a biannual conference), in particular, permutations **avoiding** certain patterns.

Let $\sigma = (p_1, p_2, \ldots, p_k)$ be a permutation of $\{1, 2, \ldots, k\}$. Then a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of $\{1, 2, \ldots, n\}$ **contains** σ provided there exists $1 \leq i_1 < i_2 < \cdots < i_k$ such that $\pi_{i_r} < \pi_{i_s}$ if and only if $p_r < p_s$. Otherwise, π **avoids** σ.

If $k = 2$ and $\sigma = (2, 1)$, then the only permutation π that avoids σ is $(1, 2, \ldots, n)$.

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Avoiding Patterns in Permutations II

What about patterns of length $k = 3$? There are 3 possibilities: $\sigma = (1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)$. Under reversal and complementation, there are only two non-equivalent: $(1,2,3)$ and $(3,1,2)$.

Examples: The permutation $(3,4,5,1,2,6,7)$ is 321-avoiding in that there does not exist a decreasing subsequence of length 3.

The permutation $(2,1,3,5,4,6)$ is 312-avoiding; no subsequence of L(arge), S(mall), M(edium).

The number of σ-avoiding permutations is the same in all cases of $k = 3$, namely,

$$C_n := \frac{\binom{2n}{n}}{n+1}, \qquad \text{the ubiquitous } n\text{th Catalan number.}$$

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312-Avoiding Patterns in Permutations; A Generalization

Let π be a permutation of $\{1, 2, \ldots, n\}$. Then π is a **312-avoiding permutation** provided π_2 has no **subsequence** a, b, c with $a > b, a > c, b < c$;

- As an $n \times n$ permutation matrix, a 312-avoiding permutation is one having no 3×3 submatrix of the form

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  Similar statements can be made for the other patterns of length 3.

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312-Avoiding (LSM-avoiding) Patterns in (0, 1)-Matrices

Examples:

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There does not exist a 312-avoiding permutation matrix $P \leq A$.

As in the examples, an $m \times n$ 312-avoiding (0,1)-matrix A contains at most $2(m+n-2)$ 1's; if A contains fewer than $2(m+n-2)$ 1's, then it is always possible to change a 0 to a 1 with the result also 312-avoiding.

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This is an upper estimate on the number of 1's a $(0,1)$-matrix A can have if it avoids a prescribed subpattern (does not have to be part of a permutation matrix $P \leq A$).

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Continuous Analogue of Permutation Matrices

These are the $n \times n$ **doubly stochastic matrices** $\mathbf{\Omega}_n$: nonnegative entries with all row and column sums equal to 1. For example,

$$\begin{bmatrix} .5 & .2 & .3 \\ .3 & .4 & .3 \\ .2 & .4 & .4 \end{bmatrix}.$$

By Birkhoff's theorem, $\mathbf{\Omega}_n$ is the convex hull of the $n \times n$ permutation matrices $\mathcal{P}_n$ and these are the extreme points.

$\dim \mathbf{\Omega}_n = (n-1)^2$, i.e, $\mathcal{P}_n$ has a linear span of dimension $(n-1)^2 + 1$.

There are several known bases of $\mathcal{P}_n$:

- (Farahat and Mirsky): the identity permutation ι_n, all 2-cycles, all 3-cycles of the form $1 \to i \to j \to 1$ where $1 < i < j \leq n$ (the $n \times n$ permutation matrices C_{1ij} with $1 < i < j \leq n$).
- We have exhibited other bases: 123-avoiding permutations, 312-avoiding permutations, bigrassmanian permutations, ...

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Continuous Analogue of ASMs

- First every ASM is a ± 1 linear combination of permutation matrices and so the dimension of the linear span of the $n \times n$ ASMs $\mathcal{A}_n$ is also $(n-1)^2 + 1$.
- Convex hull Λ_n of $\mathcal{A}_n$. **Linear characterization** is:
  All $A = [a_{ij}]$ with row and column sums equal to 1, and satisfying

  $$\sum_{j=1}^{q} a_{ij}, \sum_{j=q+1}^{n} a_{ij} \geq 0 \text{ (all } q \text{ and } i\text{) with similar inequalities for columns.}$$

- $\dim(\Lambda_n) = (n-1)^2$ and the set of extreme points of Λ_n is $\mathcal{A}_n$. Edges have been characterized,

Continuous Analogue of ASMs

- First every ASM is a ± 1 linear combination of permutation matrices and so the dimension of the linear span of the $n \times n$ ASMs $\mathcal{A}_n$ is also $(n-1)^2 + 1$.
- Convex hull Λ_n of $\mathcal{A}_n$. **Linear characterization** is:
  All $A = [a_{ij}]$ with row and column sums equal to 1, and satisfying

$$\sum_{j=1}^{q} a_{ij}, \sum_{j=q+1}^{n} a_{ij} \geq 0 \text{ (all } q \text{ and } i\text{) with similar inequalities for columns.}$$

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Higher Dimension Analogues

$n \times n \times n$ arrays with all line sums (three directions) equal to 1:

$$Permutations: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nearrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \nearrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix};$$

Really a Latin Square: $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$.

$$ASMs: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \nearrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \nearrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

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References I

- M. Bóna. *Combinatorics of Permutations*, CRC, 2004.
- H.K. Farahat, L. Mirsky, Permutation endomorphisms and refinement of a theorem of Birkhoff, *Proc. Cambridge Philos. Soc.*, 56 (1960), 322-328.
- R.E. Behrend, V.A. Knight, Higher spin alternating sign matrices, *Electron. J. Combin.* 14 (2007), 1–38.
- A. Bjőrner and F. Brenti, *Combinatorics of Coxeter Groups*, Springer 2005.
- D. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, Math. Association of America, Cambridge University Press 1999.
- R.A. Brualdi and L. Cao, Pattern-avoiding $(0, 1)$-matrices and bases of permutation matrices, *Discrete Applied Math.*, 304 (2021), 196–2011.
- R.A. Brualdi and L. Cao, Blockers of pattern avoiding permutation patterns, To be submitted soon.

References II

- R.A. Brualdi, G. Dahl, Alternating sign matrices, extensions and related cones, *Advances in Applied Mathematics*, 86 (2017), 19–49.
- R.A. Brualdi, G. Dahl, Alternating sign matrices and hypermatrices, and a generalization of latin squares, *Advances in Applied Mathematics*, 95 (2018), 116–151.
- R.A. Brualdi and G. Dahl, Alternating Sign Matrices – Extensions, Konig-properties, and Primary Sum-Sequences, *Graphs & Combinatorics*, 36 (2020), 63–92.
- R.A. Brualdi, K.P. Kiernan, S.A. Meyer, M.S. Schroeder, Patterns of Alternating Sign Matrices, *Linear Alg. Appl.* 438 (2013), 3967–3990. (**This was my initial investigation of ASMs with three students.**)
- R.A. Brualdi, H.-K. Kim, Symmetric Alternating Sign Matrices, *Australasian J. Combinatorics*, 60 (2014), 333–345.

References III

- R.A. Brualdi, H.-K. Kim, Completions of Alternating Sign Matrices, *Graphs & Combinatorics*, 31 (2015), 507–522.
- R.A. Brualdi, M.W. Schroeder, Alternating sign matrices and their Bruhat order, *Discrete Math.*, 340 (2017), 1996–2019.
- G. Kuperberg, Another proof of the alternating sign matrix conjecture, *Internat. Math. Res. Notes*, 1996, 139–150.
- W.H. Mills, D.P. Robbins, H. Rumsey, Alternating-sign matrices and descending plane partitions, *J. Combin. Theory Ser. A*, 34 (1983), 340-359.
- J. Striker, The alternating sign matrix polytope, *Electron. J. Combin.*, 16, #R41 (2009).
- D. Zeilberger, Proof of the alternating sign matrix conjecture, *Electr. J. Combinatorics*, 3 (1996), R1.