Graphs, stable permutations, and Cuntz algebra automorphisms

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December 7, 2021

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PLAN:

1. Stable permutations
2. Why stable permutations?
3. Graphs and stable permutations
4. Applications
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1. Stable Permutations

Let $n, m \in \mathbb{P}$, $u \in S_n$, $v \in S_m$, $S_n := S([n])$ (where $[n] := \{1, \ldots, n\}$, and $S(A) := \{f : A \to A : f \text{ bijection }\}$).

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Definition

The tensor product of u and v is the permutation $u \otimes v \in S([n] \times [m])$ defined by

$$(u \otimes v)(i, j) := (u(i), v(j))$$

for all $(i, j) \in [n] \times [m]$.

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Lemma

Let $u, u' \in S_n$, $v, v' \in S_m$. Then

$$(u \otimes v)(u' \otimes v') = (uu') \otimes (vv').$$

Let 1 be the identity of S_n, $u \in S([n]^r)$ and $k \in \mathbb{N}$ (where $[n]^r := [n] \times \cdots \times [n]$).

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$$\psi_k(u) := (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1})(\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes u^{-1} \otimes 1) \cdots$$

$$\cdots (u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})(1 \otimes u \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-1})$$

$$(1 \otimes 1 \otimes u \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-2}) \cdots (\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u).$$

Equivalently,

$$\psi_k(u) = (\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u^{-1})(\psi_{k-1}(u) \otimes 1)(\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u)$$

for all $k \geq 1$, where $\psi_0(u) := u^{-1}$.

Example

Let $n = 4$, $r = 2$, $k = 1$, and $u = ((1,2),(2,3))$.
Then

$$\begin{aligned}\psi_1(u)(1,2,3) &= (1 \otimes u^{-1})(u^{-1} \otimes 1)(1 \otimes u)(1,2,3) \\ &= (1 \otimes u^{-1})(u^{-1} \otimes 1)(1,1,2) \\ &= (1 \otimes u^{-1})(1,1,2) \\ &= (1,2,3)\end{aligned}$$

Example

Let $n = 9$, $r = 2$, and $u = ((1,2),(2,3),(5,9)) \in S([9]^2)$.
Then
$$\psi_4(u)(5,9,1,2,9,6) = (5,9,1,2,\underbrace{9,6}_{u})$$

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Example

Let $n = 9$, $r = 2$, and $u = ((1,2),(2,3),(5,9))$.
Then

$$\psi_4(u)(5,9,1,2,9,6) = (2,\underbrace{3,2}_{u^{-1}},3,9,6)$$

Example

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Let $u \in S([n]^r)$, $n, r \in \mathbb{P}$.

Definition

u is stable if there exists $k \in \mathbb{P}$ such that

$$\psi_k(u) \in S([n]^{r+k-1}) \otimes \{1\}$$

(i.e., if there exists $v \in S([n]^{r+k-1})$ such that $\psi_k(u) = v \otimes 1$).
In this case, the least such $k \in \mathbb{P}$ is called the rank of u.

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Remark

If u is stable of rank k then

$$\psi_k(u) = \psi_{k-1}(u) \otimes 1.$$

Example

Let $n = 4$, $r = 2$, and $u = ((1,2),(2,3))$.
Then

$$\begin{aligned}\psi_k(u)(1,1,\ldots,1,2) &= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1}) \cdots \\ &\quad \cdots (u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})(1,2,3,\ldots,3) \\ &= (2,3,\ldots,3)\end{aligned}$$

$\Rightarrow$ u is not stable.

Proposition

Let $u \in S([n]^2)$. Then u is stable of rank 1 if and only if

$$(u \otimes 1)(1 \otimes u) = (1 \otimes u)(u \otimes 1)$$

in $S([n]^3)$.

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Remark

The Yang-Baxter Equation (YBE) is

$$(u \otimes 1)(1 \otimes u)(u \otimes 1) = (1 \otimes u)(u \otimes 1)(1 \otimes u).$$

Permutations $u \in S([n]^2)$ that are solutions of the YBE are called set-theoretic solutions of the YBE, and there is quite a literature on them. No characterization of them is known.

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T. Gateva-Ivanova, P. Cameron, Multipermutation solutions of the Yang-Baxter equation, *Comm. Math. Phys.*, **309** (2012), 583-621.

Theorem (B.-Conti, 2021)

Let $(i,j),(a,b) \in [n]^2$, $(i,j) \neq (a,b)$, *and* $u := ((i,j),(a,b))$. *Then the following conditions are equivalent:*

i) *u is stable;*

ii) *u is stable of rank 1;*

iii) $\{a,i\} \cap \{b,j\} = \emptyset$.

Theorem (B.-Conti, 2021)

Let $u, v \in S([n]^2)$, u stable of rank $\leq s$, v stable of rank $\leq r$ be such that

$$(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1).$$

Then uv is stable of rank $\leq r + s$.

Theorem (B.-Conti, 2021)

Let $u, v \in S([n]^2)$*,* u *stable of rank* $\leq s$*,* v *stable of rank* $\leq r$ *be such that*

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Then uv *is stable of rank* $\leq r + s$*.*

If $u, v \in S([n]^2)$ are such that

$$(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$$

then we say that u is compatible with v (in this order).

2. Why stable permutations?

Let H be a Hilbert space and $S_1, \ldots, S_n$ be n isometries such that

$$S_i^* S_j = \delta_{i,j} 1$$

and

$$\sum_{i=1}^{n} S_i S_i^* = 1.$$

The Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra generated by

$$\{S_1, \ldots, S_n, S_1^*, \ldots, S_n^*\}.$$

It is well known that there is a bijective correspondence $u \mapsto \lambda_u$ between the unitary elements $u \in \mathcal{O}_n$ (an element $u \in \mathcal{O}_n$ is unitary if $uu^* = u^*u = 1$) and the endomorphisms of $\mathcal{O}_n$, where

$$\lambda_u(S_i) := uS_i,$$

for $i = 1, \ldots, n$, and whose inverse maps the endomorphism λ to the unitary

$$u_\lambda := \sum_{i=1}^{n} \lambda(S_i) S_i^* .$$

However, the problem of deciding if a given endomorphism λ_u is actually an automorphism is extremely complicated and no general procedure is known to date. For this reason, producing examples is definitely important. On the other hand, solutions of this problem for specific classes of endomorphisms have been found.

Among the early examples, in 1992 Matsumoto and Tomiyama found an (outer) automorphism of $\mathcal{O}_4$ mapping

$$S_1 \mapsto S_2$$

$$S_2 \mapsto S_3$$

$$S_3 \mapsto S_2(S_1S_3^* + S_3S_1^* + S_2S_4^* + S_4S_2^*)$$

and

$$S_4 \mapsto S_4(S_1S_3^* + S_3S_1^* + S_2S_4^* + S_4S_2^*).$$

This automorphism is now simply understood as an element of the reduced Weyl group of $\mathcal{O}_4$. The definition of these reduced Weyl groups goes back to Cuntz, although the terminology was introduced much later by Conti and Szymanski.

For $\alpha = (\alpha_1, \dots, \alpha_k) \in [n]^k$ we define $S_\alpha := S_{\alpha_1} \cdots S_{\alpha_k}$. The elements of the form

$$\{S_\alpha S_\beta^* \mid \alpha, \beta \in [n]^k\}$$

span a subalgebra $\mathcal{F}_n^k$ of $\mathcal{O}_n$, isomorphic to the $*$-algebra of complex matrices M_{n^k}, which is further isomorphic to $M_n \otimes \cdots \otimes M_n$ (k factors), in such a way that $S_\alpha S_\beta^*$ corresponds to the matrix that has entry 1 in position α, β and 0 elsewhere.

As $\mathcal{F}_n^k \subset \mathcal{F}_n^{k+1}$, taking the limit for $k \to \infty$ one obtains the inductive limit C^*-algebra $\mathcal{F}_n \subset \mathcal{O}_n$, also known as the core UHF subalgebra, isomorphic to the infinite tensor product $M_n \otimes M_n \otimes \cdots$. Further, the endomorphism

$$\varphi(x) := \sum_{i=1}^{n} S_i x S_i^*$$

of $\mathcal{O}_n$ restricts to an endomorphism of $\mathcal{F}_n$ which, in the infinite tensor product picture, corresponds to the tensor shift map $x \mapsto 1_{M_n} \otimes x$.

Inside $\mathcal{F}_n^k$ one has the C^*-subalgebra $\mathcal{D}_n^k$ generated by the family of orthogonal projections

$$\{S_\alpha S_\alpha^* \mid \alpha \in [n]^k\},$$

and isomorphic to the algebra of diagonal matrices in M_{n^k}. Again, $\mathcal{D}_n^k \subset \mathcal{D}_n^{k+1}$ and the corresponding inductive limit $\mathcal{D}_n$ is a commutative C^*-subalgebra of $\mathcal{F}_n$ (and $\mathcal{O}_n$), which turns out to have the Cantor set $[n]^\infty$ as Gelfand spectrum.
Now, following the insight by Cuntz, the reduced Weyl group of $\mathcal{O}_n$ can be defined as

$$\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)/\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$$

where $\mathrm{Aut}(\mathcal{O}_n, X)$ is the subset of $\mathrm{Aut}(\mathcal{O}_n)$ consisting of the automorphisms which leave X invariant, while $\mathrm{Aut}_X(\mathcal{O}_n)$ is the set of those which fix X pointwise.

With some more work the reduced Weyl group can be further identified with the set of automorphisms λ_u induced by the so-called permutative unitaries in $\cup_{k\geq 1}\mathcal{F}_n^k$. Moreover, for unitaries u in $\mathcal{F}_n^k$, for any k, it was shown in Conti-Szymanski that λ_u is an automorphism precisely when the sequence of unitaries

$$(\varphi^r(u^*)\cdots\varphi(u^*)u^*\varphi(u)\cdots\varphi^r(u))_{r\geq 0}$$

eventually stabilizes. By identifying permutative unitaries in $\mathcal{F}_n^k$ with permutation matrices in M_{n^k} and thus with permutations of the set $\{1,\ldots,n^k\}$ and finally (by lexicographic ordering) with permutations of the set $[n]^k$, we are thus lead to the concept of stable permutations which we have defined. These permutations precisely label the elements of the restricted Weyl group.

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Theorem

$\mathcal{O}_n$ is isomorphic to a subalgebra of $\mathcal{O}_2$, for all $n \geq 2$.

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Theorem

$$\mathcal{O}_2 \otimes \mathcal{O}_2 \approx \mathcal{O}_2.$$

Here are some references:

J. Cuntz, Simple C*-algebras generated by isometries. *Comm. Math. Phys.* **57** (1977), no. 2, 173-185.

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R. Conti, J. H. Hong, W. Szymański, The Weyl group of the Cuntz algebra. *Adv. Math.* **231** (2012), no. 6, 3147-3161.

3. Graphs and stable permutations

Let G be a directed graph on vertex set $[n]^{t-1}$ ($t \in \mathbb{P}$, $t \geq 2$) and $u \in S([n]^t)$. Define two directed graphs $\mathcal{R}_u(G)$ and $\mathcal{L}_u(G)$ as

follows. If

$$(a_1, \dots, a_{t-1}) \to (b_1, \dots, b_{t-1})$$

in G, and $z \in [n]$ then

$$F_1(u(a_1, \dots, a_{t-1}, z)) \to F_1(u(b_1, \dots, b_{t-1}, z))$$

in $\mathcal{R}_u(G)$, where $F_i : [n]^t \to [n]^{t-1}$ is defined by

$$F_i(x_1, \dots, x_t) := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_t)$$

Similarly, if

$$(a_1, \ldots, a_{t-1}) \to (b_1, \ldots, b_{t-1})$$

in G, and $z \in [n]$ then

$$F_t(u(z, a_1, \ldots, a_{t-1})) \to F_t(u(z, b_1, \ldots, b_{t-1}))$$

in $\mathcal{L}_u(G)$.

Now define, for $u \in S([n]^t)$, a graph $\Gamma_0(u)$ on vertex set $[n]^{t-1}$ by letting

$$(a_1, \ldots, a_{t-1}) \to (b_1, \ldots, b_{t-1})$$

if and only if there are $w, z \in [n]$ such that

$$u^{-1}(z, a_1, \ldots, a_{t-1}) = (w, b_1, \ldots, b_{t-1})$$

Similarly, define a graph $\Gamma_0^{\#}(u)$ on vertex set $[n]^{t-1}$ by letting

$$(a_1, \ldots, a_{t-1}) \to (b_1, \ldots, b_{t-1})$$

if and only if there are $w, z \in [n]$ such that

$$u^{-1}(a_1, \ldots, a_{t-1}, z) = (b_1, \ldots, b_{t-1}, w)$$

Theorem (B.-Conti-Nenashev, 2021)

Let $u \in S([n]^t)$, $t > 1$. Then u is stable if and only if there is $M \in \mathbb{P}$ such that

$$(\mathcal{L}_u)^M(\Gamma_0^{\#}(u))$$

and

$$(\mathcal{R}_{u^{-1}})^M(\Gamma_0(u))$$

consist only of loops.

Proof.

(Sketch)

We start by defining two numbers.

- $N(u)$ is the least integer such that, for all $k \geq N(u)$

$$\psi_k(u)$$

  does not change the last $t-1$ coordinates. If there is no such integer, we set $N(u) = +\infty$.

- $N^{\#}(u)$ is the least integer such that, for all $k \geq N^{\#}(u)$,

$$\Pi_k(u) := (u \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k}) \cdots (\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes u \otimes 1) \times (\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u^{-1}) \cdots (u^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k}).$$

  does not change the first $t-1$ coordinates. If there is no such integer, we set $N^{\#}(u) = +\infty$.

□

Theorem

Let $u \in S([n]^t)$, $t > 1$, *then* u *is stable if and only if both* $N(u)$ *and* $N^{\#}(u)$ *are finite, and in this case*

$$\max\{N(u) - t + 2, N^{\#}(u) - t + 2\} \leq \mathrm{rk}(u) \leq N(u) + N^{\#}(u) + t - 1.$$

We now introduce two sequences $\Delta_k(u), \Delta_k^{\#}(u), k \geq 0$ of simple directed graphs. For $k \geq 0$ we define $\Delta_k(u)$ and $\Delta_k^{\#}(u)$, as follows:

- $V(\Delta_k(u)) = V(\Delta_k^{\#}(u)) = [n]^{t-1}$;
- given $(a_1, a_2, \ldots, a_{t-1}), (b_1, b_2, \ldots, b_{t-1}) \in [n]^{t-1}$ there is a directed edge

$$(a_1, a_2, \ldots, a_{t-1}) \to (b_1, b_2, \ldots, b_{t-1})$$

  in $\Delta_k(u)$ if

$$\psi_k(u)(*, \ldots, *, a_1, a_2, \ldots, a_{t-1}) = (*, \ldots, *, b_1, b_2, \ldots, b_{t-1})$$

- given $(a_1, a_2, \ldots, a_{t-1}), (b_1, b_2, \ldots, b_{t-1}) \in [n]^{t-1}$ there is a directed edge

$$(a_1, a_2, \ldots, a_{t-1}) \to (b_1, b_2, \ldots, b_{t-1})$$

in $\Delta_k^{\#}(u)$ if

$$\Pi_k(u)(a_1, a_2, \ldots, a_{t-1}, *, \ldots, *) = (b_1, b_2, \ldots, b_{t-1}, *, \ldots, *)$$

where

$$\Pi_k(u) := (u \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k}) \cdots (\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes u \otimes 1) \times$$
$$(\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u^{-1}) \cdots (u^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k}).$$

Note that these graphs all have the same vertex set. Also, if a permutation u is stable, then $\Delta_a(u)$ and $\Delta_b^{\#}(u)$ consist only of loops if $a \geq N(u)$ and $b \geq N^{\#}(u)$. Actually, $N(u)$ is the least integer such that $\Delta_k(u)$ consists only of loops for all $k \geq N(u)$, and similarly for $N^{\#}(u)$.

Proposition

Let $u \in S([n]^t)$ and $k \geq 0$. If the graph $\Delta_k(u)$ has a directed path from $(a_1, a_2, \ldots, a_{t-1})$ to $(b_1, b_2, \ldots, b_{t-1})$, then $\Delta_k(u)$ also has a directed path from $(b_1, b_2, \ldots, b_{t-1})$ to $(a_1, a_2, \ldots, a_{t-1})$. So, strongly connected components and connected components coincide for the graphs $\Delta_k(u)$. Similarly for $\Delta_k^{\#}(u)$.

Theorem

Let $u \in S([n]^t)$. Then

$$\Delta_{k+1}(u) = \mathcal{R}_{u^{-1}}(\Delta_k(u))$$

and

$$\Delta^{\#}_{k+1}(u) = \mathcal{L}_u(\Delta^{\#}_k(u))$$

for all $k \geq 0$.

Therefore

$$\Delta_k(u) = (\mathcal{R}_{u^{-1}})^k(\Gamma_0(u))$$

and

$$\Delta^{\#}_k(u) = (\mathcal{L}_u)^k(\Gamma^{\#}_0(u))$$

for all $k \geq 0$, and the result follows.

Example

$n = 9,\ u = ((2,1),(2,3),(4,3))\,((8,9),(2,9))$

6. Applications

Let

$$N_n^r := |\{u \in S([n]^r) : \ u \text{ stable } \}|$$

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n	r	N_n^r
1	2	1
2	2	4
3	2	576
4	2	10020

Corollary (B.-Conti-Nenashev, 2021)

Let $n, r \in \mathbb{P}$, $r \geq 2$. Then

$$\lim_{r\to\infty} \frac{N_n^r}{(n^r)!} = 0$$

and

$$\lim_{n\to\infty} \frac{N_n^r}{(n^r)!} = 0.$$

Proof.

(Sketch)

1. There are almost no permutations u such that $\Gamma_0(u)$ is disconnected.
2. There are almost no permutations u such that $\mathcal{R}_{u^{-1}}(\Gamma_0(u))$ is disconnected.
3. If $\Gamma_0(u)$ and $\mathcal{R}_{u^{-1}}(\Gamma_0(u))$ are connected then $(\mathcal{R}_{u^{-1}})^M(\Gamma_0(u))$ is connected for all $M \in \mathbb{P}$.

□

Let $\mathcal{C}_i(r)$ be the set of all strict compositions of r into i parts ($r \in \mathbb{N}$, $i \in \mathbb{P}$).

Theorem (B.-Conti-Nenashev, 2021)

Let $r, n \in \mathbb{N}$. Then the number of r-cycles in $S([n]^2)$ that are stable of rank 1 is

$$(r-1)! \sum_{i=1}^{r} \binom{n}{i} \sum_{(a_1,\dots,a_i)\in\mathcal{C}_i(r)} \prod_{j=1}^{i} \binom{n-i}{a_j}.$$

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In particular, if one chooses an r-cycle uniformly at random in $S([n]^2)$, then the probability that this is stable goes to 1 as $n \to \infty$.

Let $u := ((a_1, b_1), \dots, (a_r, b_r))$ be an r-cycle in $S([n]^2)$. Let $S^+(u)$ be the directed graph on vertex set $[r]$ define by letting

$$i \to j \iff a_i = b_j \text{ or } a_{i+1} = b_j$$

Theorem (B.-Conti-Nenashev, 2021)

Let $u \in S([n]^2)$ be an r-cycle. Then

$$S^+(u) \text{ acyclic } \Rightarrow u \text{ stable.}$$

Theorem (B.-Conti-Nenashev, 2021)

Let $u := ((a_1, b_1), \dots, (a_r, b_r)) \in S([n]^2)$ be an r-cycle, $r \leq 4$. Then TFAE:

i) *u is stable;*

ii) *u is stable of rank $\leq r$;*

iii) *$S^+(u)$ is acyclic;*

iv) *u is a compatible product of stable transpositions.*

For $u := ((a_1, b_1), \dots, (a_r, b_r)) \in S([n]^2)$ an r-cycle let

$$S(u) := \{(i,j) \in [r]^2 : a_i = b_j\}.$$

Theorem (B.-Conti-Nenashev, 2021)

Let $u \in S([n]^2)$ be a 5-cycle. Then TFAE:

i) *u is stable;*

ii) *either $S^+(u)$ is acyclic or*

$$S(u) = \{(1,2),(1,3),(3,5),(5,2),(5,3)\} + (j,j)$$

for some $j \in [5]$ (sum modulo 5).

7. Open Problems

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It is known that N_n^r is always divisible by $(n^{r-1})!$.

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Problem

Compute

$$|\{u \in S([n]^2) : (1 \otimes u)(u \otimes 1) = (u \otimes 1)(1 \otimes u)\}|$$

and

$$|\{u \in S([n]^2) \ : \ (1 \otimes u^{-1})(u^{-1} \otimes 1)(1 \otimes u) = (u \otimes 1)\}|$$

for all $n \in \mathbb{N}$.

In the case of permutations in $S_n \otimes S_n$ these problems can be solved.

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$$|\{u \in S_n \otimes S_n : \ (1 \otimes u)(u \otimes 1) = (u \otimes 1)(1 \otimes u)\}|$$

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so this number is t_n^2 where t_n is the number of involutions in S_n (equivalently, the number of standard Young tableaux of size n).

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For the case of two horizontal stable transpositions we have the following conjecture.

Conjecture

Let $(a_1, b_1), (a_1, b_2), (a_2, b_3), (a_2, b_4) \in [n]^2$ *be distinct,* $a_1 \neq a_2$*, and*

$$u := \left((a_1, b_1), (a_1, b_2)\right) \left((a_2, b_3), (a_2, b_4)\right).$$

Then u *is stable of rank 1 if and only if either*

$$\{a_1, a_2\} \cap \{b_1, b_2, b_3, b_4\} = \emptyset$$

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The conjecture holds if $n \leq 8$. Also, our results imply that if either condition holds then u is stable of rank 1.

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So, the following seem like natural questions to investigate.

Problem

Is the reduced Weyl group of $\mathcal{O}_n$ generated by

$$\left\{ \lambda_t : t \in \bigcup_{r \in \mathbb{N}} S([n]^r),\ t \text{ stable transposition} \right\} ?$$

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Problem

Is the reduced Weyl group of $\mathcal{O}_n$ generated by involutions?