MSU 2 March 2022

Tuesday, March 1, 2022 6:22 AM

Facts of symmetric edge polytopes: Maximizing families and experimental results

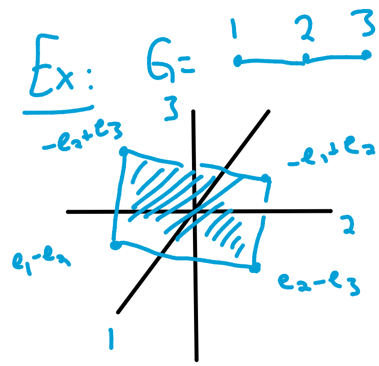
joint w/ Kaitlin Bruegge

$G=(V,E)$ a finite simple graph.

$P_G := \mathrm{conv}\{\pm(e_i-e_j) : ij\in E\}$

↑ <u>symmetric edge polytope</u>

Ex: $G=$



<u>Notes:</u> $P_G \subseteq \{\vec{x}\in\mathbb{R}^V : \mathbb{1}\cdot\vec{x}=0\}$.

$\vec{0}\in P_G^{\circ}$.

<u>Connection to Kuramoto models</u> (Chen, Davis, Mehta)

Kuramoto model

https://en.wikipedia.org/wiki/Kuramoto_model

From Wikipedia, the free encyclopedia

The **Kuramoto model** (or **Kuramoto–Daido model**), first proposed by Yoshiki Kuramoto (蔵本 由紀, *Kuramoto Yoshiki*),[1][2] is a mathematical model used to describe synchronization. More specifically, it is a model for the behavior of a large set of coupled oscillators.[3][4] Its formulation was motivated by the behavior of systems of chemical and biological oscillators, and it has found widespread applications in areas such as neuroscience[5][6][7][8] and oscillating flame dynamics.[9][10] Kuramoto was quite surprised when the behavior of some physical systems, namely coupled arrays of Josephson junctions, followed his model.[11]

The model makes several assumptions, including that there is weak coupling, that the oscillators are identical or nearly identical, and that interactions depend sinusoidally on the phase difference between each pair of objects.

For G, we have the following system: $N_G(i)=\{j : ij\in E\}$.

$$\frac{d\theta_i}{dt} = \omega_i - \sum_{j\in\mathcal{N}_G(i)} a_{ij}\sin(\theta_i-\theta_j) \quad \text{for } i=0,\dots,n$$

Setting each of these to 0 and making a change of variables gives

$$\omega_i - \sum_{j \in \mathcal{N}_G(i)} a'_{ij} \left(\frac{x_i}{x_j} - \frac{x_j}{x_i} \right) = 0 \quad \text{for } i = 1, \ldots, n$$

<u>Thm (Davis, Chen, Mehta)</u>: The number of isolated $\mathbb{C}^*$ solutions to this system of equations is at most the (normalized) volume of P_G.

Connection to Ehrhart Theory:

P_G is a lattice polytope, i.e., $\operatorname{conv}\{v_1, \ldots, v_n\}$ for $v_i \in \mathbb{Z}^d$.

Results of Ehrhart and Stanley imply that if G is connected,

$$\sum_{t \geq 0} |t \cdot P_G \cap \mathbb{Z}^d| z^t = \frac{\sum_{j=0}^{d-1} h_j^* z^j}{(1-z)^d} \quad \text{for some } h_j^* \in \mathbb{Z}_{\geq 0} \text{ and } h_0^* = 1.$$

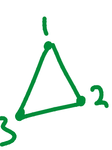
$(h_0^*, h_1^*, \ldots, h_{d-1}^*)$ is the Ehrhart h^*-vector of P_G.

<u>Thm (Higashitani)</u>: P_G is <u>reflexive</u>, i.e., the dual of P_G in its affine span is a lattice polytope.

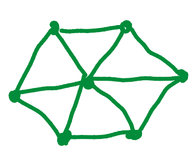
<u>Thm (Hibi)</u>: P reflexive $\Rightarrow$ h_p^* is symmetric.

<u>Thm (Higashitani, Jochemko, Michałek)</u>: P_G admits a regular unimodular triangulation.

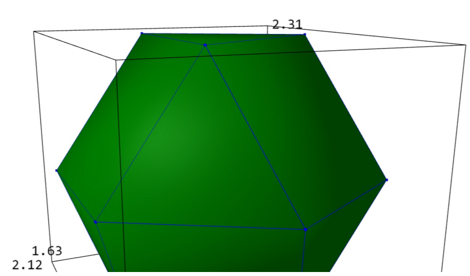
<u>Cor (via a result of Bruns-Römer)</u>: The h^*-vector of P_G is unimodal.

<u>Ex</u>: $K_3 =$ 

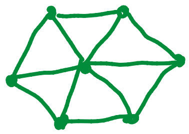
$h^*_{P_{K_3}} = (1, 4, 1)$

$P_{K_3} =$ 

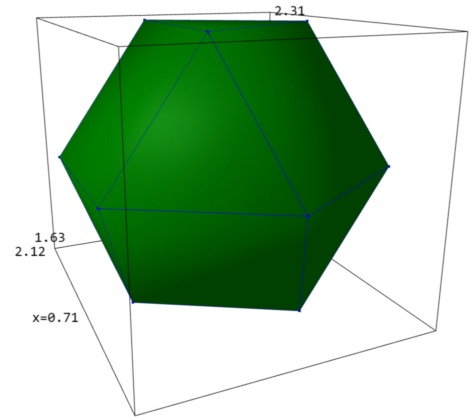
<u>Ex</u>: $G = K_4$

$P_{K_4} =$ 

K_3

$P_{K_3} =$ 

<u>Ex:</u> $G = K_4$

$P_{K_4} =$ 

$h^*_{P_{K_4}} = (1,9,9,1)$

In both of these areas, the facet structure of P_G is important.

<u>Our goal:</u> Study facet numbers of P_G in relation to G.

<u>Def$^{\underline{n}}$</u>: Let $N(G) = \#$ of facets of P_G.

Thm (Higashitani, Jochemko, Michałek) + Chen, Davis, Korchevskaia: If G is connected, then Facets of P_G are of the form

$$\sum_{i \in V(G)} f(i) \cdot x_i = 1$$

where $f: V \to \mathbb{Z}$ is defined up to $f - g =$ constant and

1) if $ij \in E$, $|f(i) - f(j)| \leq 1$

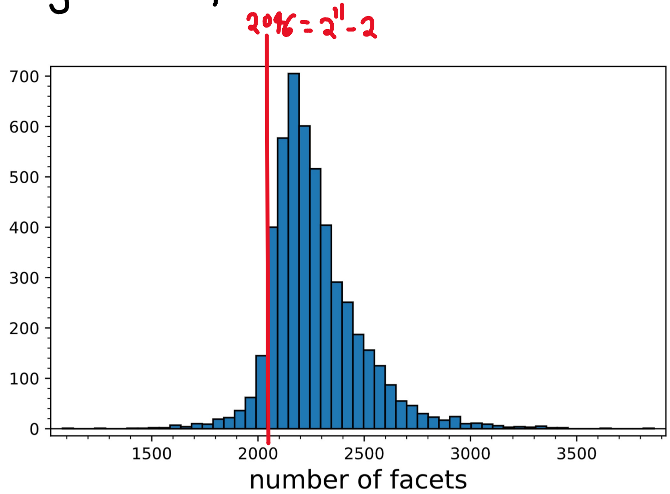
and 2) The edge-induced subgraph $E_f = \{ij \in E : |f(i) - f(j)| = 1\}$ is a maximal connected spanning bipartite subgraph of G.

<u>Ex:</u> $N(K_n) = 2^n - 2$. Why? Any nontrivial subset A of $[n]$ induces a 0-1 labeling of $[n]$ where $f(i) = 0$ for $i \in A$ and $f(i) = 1$ for $i \in A^c$. These are the facets since E_f satisfies (2).

<u>Q:</u> What might we expect the facet number to be?

$2046 = 2^{11} - 2$

Q: What might we expect the facet number to be:



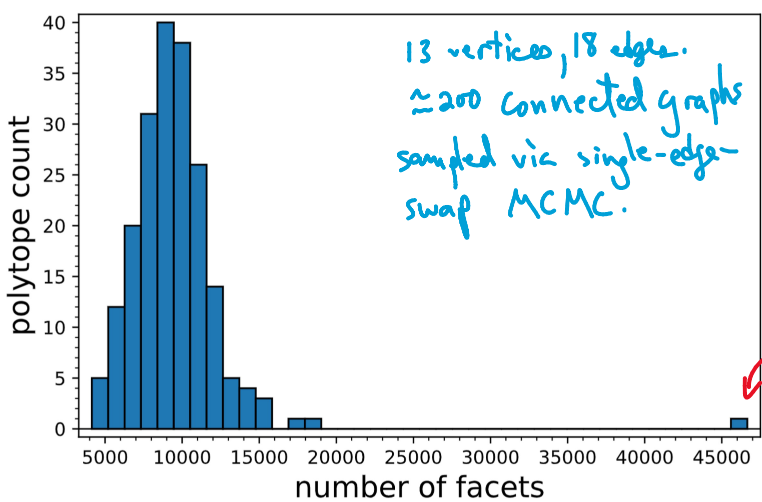
A: Experimentally, we expect more facets than P_{K_n} for a connected G.

Erdös-Renyi model.

FIGURE 2. Histogram of $N(G)$ for 4874 connected graphs sampled from $G(11, 0.45)$.

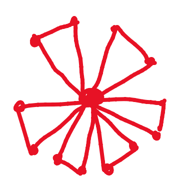
Q: What if we fix the number of edges?

A: Complicated....



What is this?!?

This has $6^6 = 46{,}656$ facets.

$G =$ 

Conjecture (B., Bruegge): For connected graphs on n vertices with $\frac{3(n-1)}{2}$ edges, these "windmill" graphs maximize $N(G)$, with value $6^{n-1/2}$.

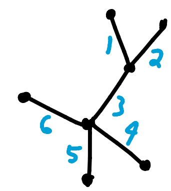
Question: For graphs w/ a fixed number of vertices and edges, which graphs maximize $N(G)$?

Def^n: G, H simple graphs. $G \vee H$ ("wedge") is obtained by identifying G & H at a vertex.

Prop$^{\underline{n}}$ (Ohsugi, Tsuchiya): $P_{G \vee H} = P_G \oplus P_H$.
↑ free sum

Cor: $N(G \vee H) = N(G) \cdot N(H)$.

Cor: If T is a tree on n vtcs, then $N(T) = 2^{n-1}$.



Wedge edges sequentially

Lemma (Chen-Davis-Mehta, Nill)

For any m, $N(C_m) = \begin{cases} \binom{m}{m/2} & m \text{ even} \\ m \cdot \binom{m-1}{m-1/2} & m \text{ odd} \end{cases}$

Thm (B., Bruegge): If G is connected & has n vtcs and n edges, $N(G)$ is maximized by C_n if n is odd and by $C_{n-1} \vee e$ if n is even.

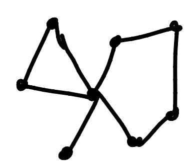
Take-away: "Make a big odd cycle".

Q: What about n vtcs and $n+1$ edges? (Surprisingly hard!)

For $n \geq 3$, let $M(n)$ be the number of facets of P_G where

$$G := \begin{cases} C_{k+1} \vee C_{k-1} & n = 2k-1, \ k \text{ even} \\ C_k \vee C_k & n = 2k-1, \ k \text{ odd} \\ C_{k+1} \vee C_{k-1} \vee e & n = 2k, \ k \text{ even} \\ C_k \vee C_k \vee e & n = 2k, \ k \text{ odd} \end{cases}$$

Ex: $n = 8 = 2 \cdot 4$



Conjecture (B., Bruegge):

For any connected graph G with n vertices and $n+1$ edges, we have

$$N(G) \leq M(n) .$$

Two cases:

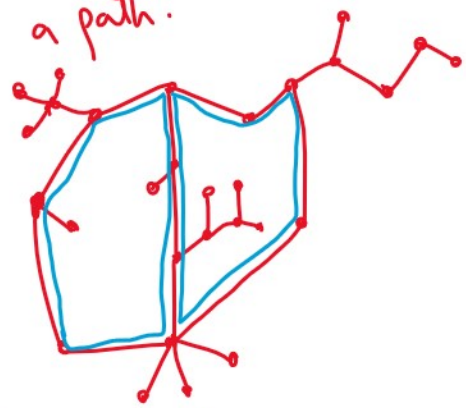
2) G has 2 cycles sharing

Two cases:

A) G has 2 edge-disjoint cycles.

We've proved it in this case.

B) G has 2 cycles sharing a path.



This case is still open.

Lemma (B., Brugge) We can assume G has no leaves, since

$$2 \cdot M(n) \leq M(n+1).$$

Thus, we reduce to this case:

Definition 3.11. For positive integers x_1, x_2, x_3 with $\sum x_i = n+1$, let $CB(x_1, x_2, x_3)$ denote the graph on n vertices and $n+1$ edges constructed by identifying the endpoints on three paths having x_1, x_2, and x_3 edges (respectively).

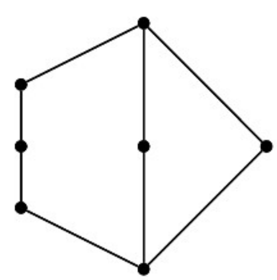


FIGURE 3. $CB(4, 2, 2)$

Definition 3.12. Let x_1, x_2, x_3 be positive integers that are either all even or all odd. We define the function $F(x_1, x_2, x_3)$ that first sorts the inputs so that without loss of generality $x_1 \geq x_2 \geq x_3$, then assigns the value

$$F(x_1, x_2, x_3) := \sum_{j=0}^{x_3} \binom{x_3}{j} \binom{x_2}{\frac{1}{2}(x_2 - x_3) + j} \binom{x_1}{\frac{1}{2}(x_1 - x_3) + j}.$$

Ex:

1 4 6 4 1

1 6 15 20 15 6 1

1 8 28 56 70 56 28 8 1

1 6

1 8 28 56 70 56 28 8 1

multiply columns and add.

(B., Bruegge)

Theorem 3.13. *The number of facets of the symmetric edge polytope for $CB(x_1, x_2, x_3)$ is computed as follows.*

(i) *For x_1, x_2, x_3 either all even or all odd,*

$$N(CB(x_1, x_2, x_3)) = F(x_1, x_2, x_3)\,.$$

(ii) *For o_1, o_2 odd, e_1 even, and all at least 2,*

$$N(CB(o_1, o_2, e_1)) = e_1 F(o_1, o_2, e_1 - 1) + o_1 o_2 F(o_1 - 1, o_2 - 1, e_1)\,.$$

For e_1, e_2 even, o_1 odd, and all at least 2,

$$N(CB(e_1, e_2, o_1)) = o_1 F(e_1, e_2, o_1 - 1) + e_1 e_2 F(e_1 - 1, e_2 - 1, o_1)\,.$$

(iii) *For e_1, e_2 even, and $o_1 = 1$,*

$$N(CB(e_1, e_2, 1)) = e_1 e_2 F(e_1 - 1, e_2 - 1, 1) + N(C_{e_1} \vee C_{e_2})$$

(iv) *For e_1 even, $o_1 \geq 3$ odd,*

$$N(CB(e_1, o_1, 1)) = e_1 F(e_1 - 1, o_1, 1) + o_1 N(C_{o_1 - 1} \vee C_{e_1})$$

(v) *For e_1 even,*

$$N(CB(e_1, 1, 1)) = e_1 F(e_1 - 1, 1, 1) + N(C_{e_1})$$

Thm (B., Bruegge): If x_1, x_2, x_3 have same parity, w/ $x_1 + x_2 + x_3 = n+1$, then

(★) $N(CB(x_1, x_2, x_3)) = F(x_1, x_2, x_3) \leq M(n)$.

Pf is a lot of Stirling formula estimates and inequalities.

Note: It is still open to prove (★) in the different parity case.

An aside about $F(x_1, x_2, x_3)$:

Conjecture: If $\{x_i\}$ all even or all odd, $x_1 \geq x_2 \geq x_3$,

$$F(x_1, x_2, x_3) \leq F(x_1 + 2, x_2, x_3 - 2)$$

and $F(x_1, x_2, x_3) \leq F(x_1 + 2, x_2 - 2, x_3)$ (when relative order is preserved)

We've checked this for all $x_1 + x_2 + x_3 = n+1$ up to $n = 399$.

General Q: What properties of G lead to many facets?

One empirically-supported answer: average local clustering.

Defn: G finite simple. For $i \in V$, $N_G(i) = \{j : ij \in E\}$, and
$NE_G(i) = \{jk : jk, ij, ik \in E\}$.

The Watts-Strogatz local clustering of G is

$$C_{WS} = \frac{1}{|V|} \cdot \sum_{i \in V} \frac{|NE_G(i)|}{\binom{|N_G(i)|}{2}}.$$

Erdős-Renyi Sampling:

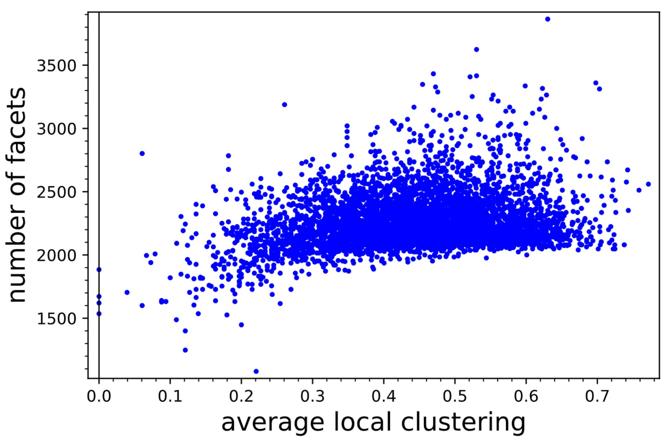


FIGURE 1. Data from a sample of 4874 connected graphs sampled from $G(11, 0.45)$.

Single-edge-swap MCMC:

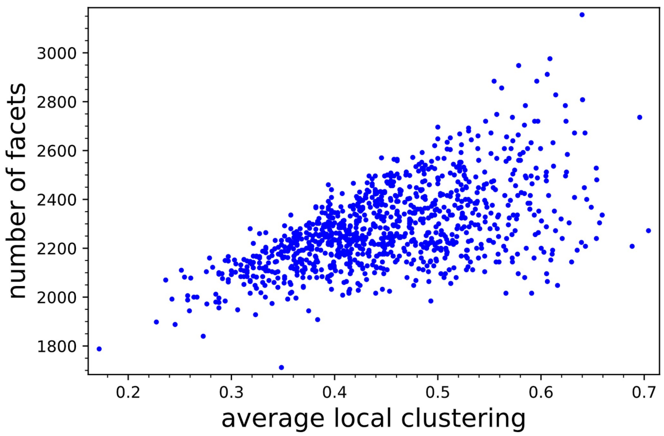


FIGURE 3. Data from a sample of 1001 graphs with 11 vertices and 25 edges.

Double-edge-swap MCMC for fixed degree sequence:

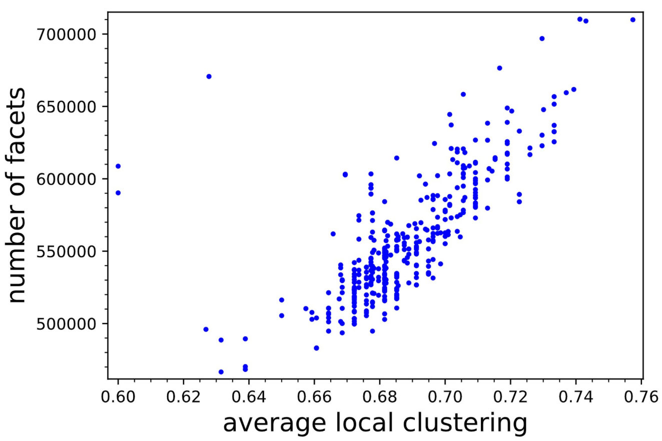


FIGURE 5. Data from 370 graphs having 18 vertices and degree sequence $[3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 16, 16]$.

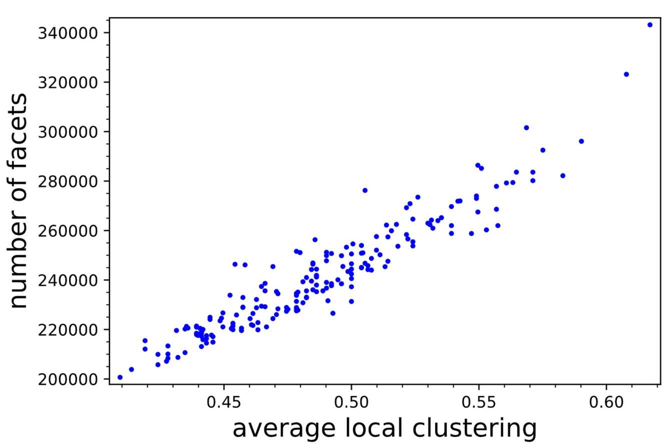
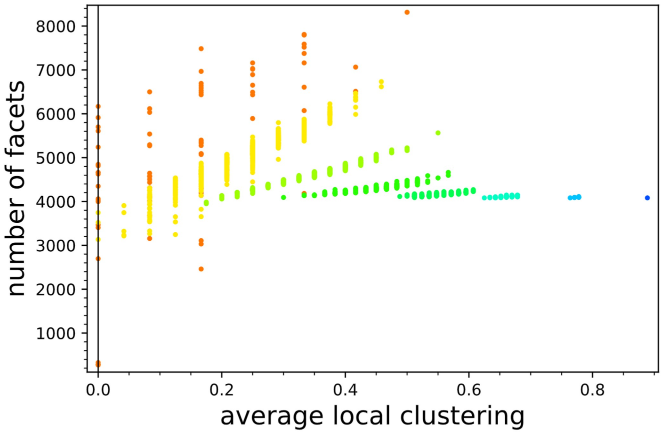


FIGURE 9. The plot shows the average local clustering and number of facets for a sample of 192 graphs with 17 vertices and degree sequence $\{3, 3, 3, 4, 4, 4, 4, 4, 4, 4, 4, 4, 5, 5, 5, 5, 15\}$.



k-regular: smaller k on left, larger to the right.

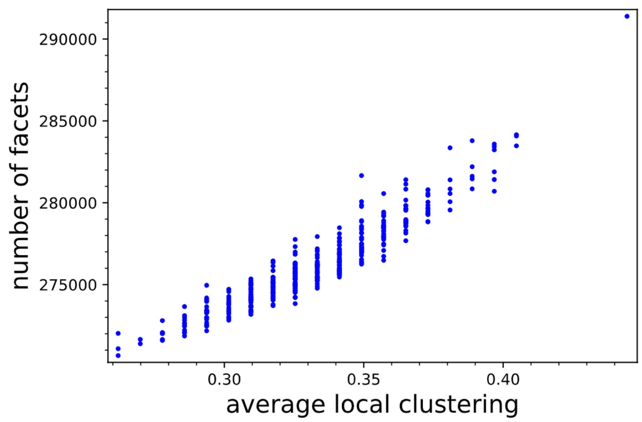


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Figure 11. Data from a sample of 399 connected graphs on 18 vertices with degree sequence [7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7] obtained by MCMC using double-edge swaps.

Thank you!