Polynomial Generalized Permutation Classes

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MSU

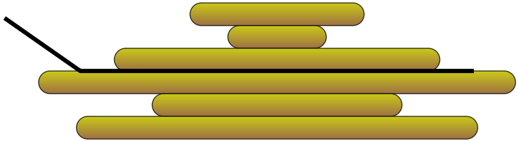
Motivation: The pancake problem

The original statement

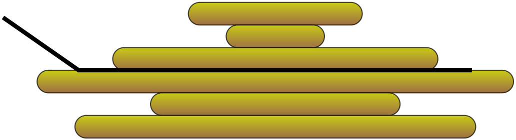
The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?

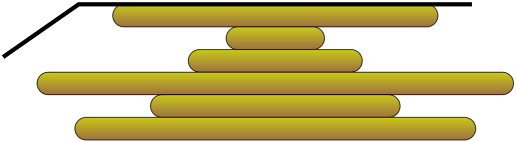
-Harry Dweighter, December 1975. American Math. Monthly

Pancake flip illustration



Pancake flip illustration





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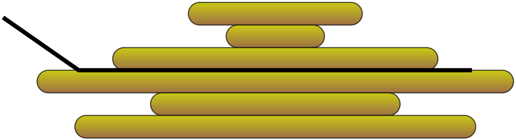

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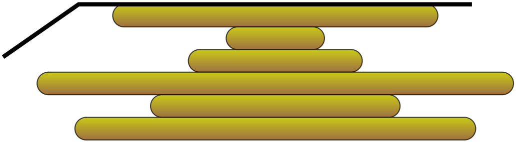
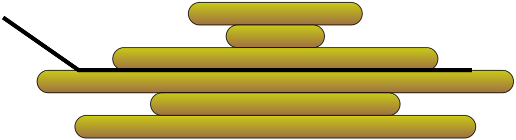


Pancake flip illustration



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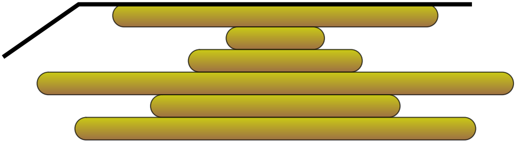
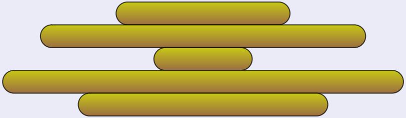


Figure: 412635

Largest pancake first algorithm

Example



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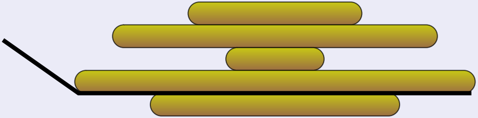

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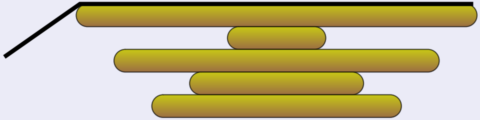

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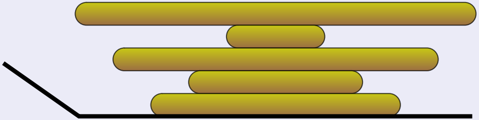

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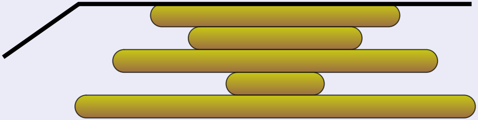

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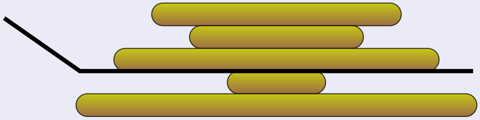

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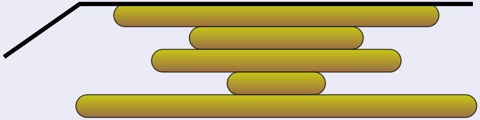

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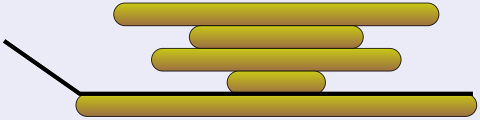

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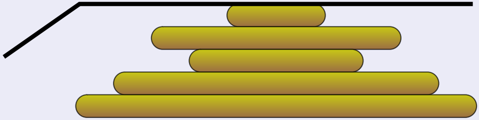

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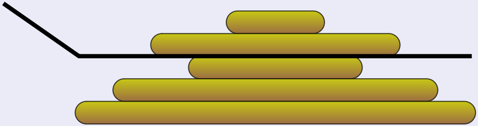
Largest pancake first algorithm

Example



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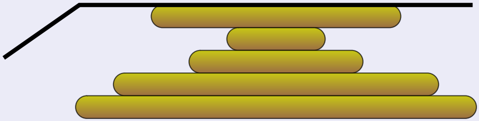
Largest pancake first algorithm

Example



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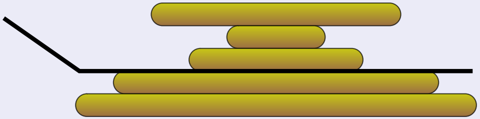

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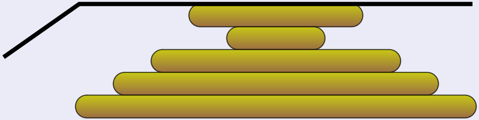

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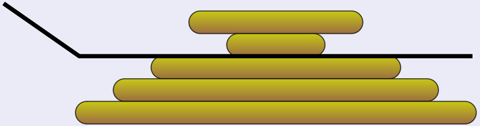

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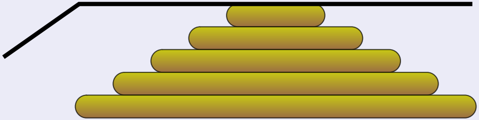

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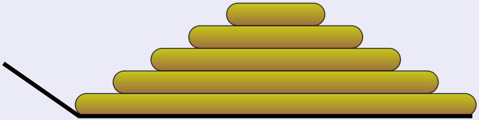

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This took **7** flips!

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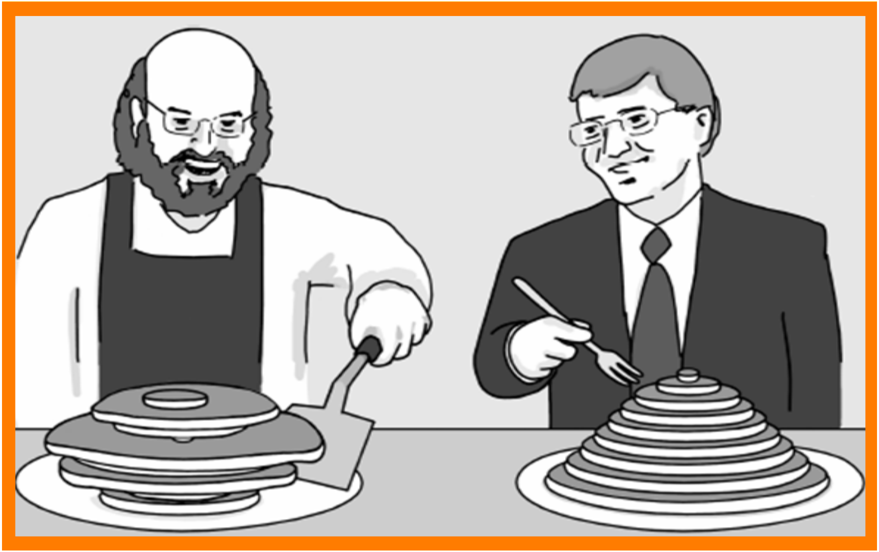
- Pancake sorting is NP-hard: Laurent Bulteau, et al. (2015). To be specific, finding the optimal sequence of flips to sort a stack is NP-hard



Source: Neil Jones and Pavel Pevzner,
2004 “Introduction to BioInformatics Algorithms”

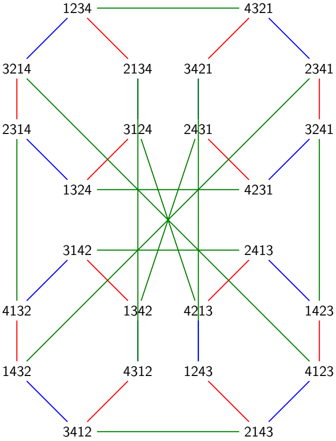
Pancake graphs

We can think of these pancake flips as *prefix-reversal generators* of the symmetric group S_n

$$r_i = i\ (i-1) \cdots 1\ (i+1)\ (i+2) \cdots n,$$

with $2 \leq i \leq n$.

Pancake graph P_4



How many stacks of n pancakes take exactly k flips to be sorted?

A great deal is known about the structure of cycles in P_n. In particular, there is only one type of 6-cycle in the graph and no cycles shorter than length 6 exist.

1. If $k = 0$, then 1.
2. If $k = 1$, then $n - 1$.
3. If $k = 2$, then $(n-1)(n-2)$.
4. If $k = 3$, then $(n-1)(n-2)^2 - 1$ (the "-1" comes from the one 6-cycle)

How many stacks of n pancakes take exactly 4 flips to be sorted?

Theorem

(B., Buehrle and Patidar, 2019) If $n \geq 3$, there are

$$\frac{1}{2}(2n^4 - 15n^3 + 29n^2 + 6n - 34)$$

stacks of n pancakes that take exactly 4 flips to be sorted.

The proof used elementary methods like the our classification of 7- and 8-cylces in P_n and PIE. It was a lot of book keeping.

There's gotta be a better way!

The Homberger–Vatter Algorithm (2016)

Vince Vatter emailed us to tell us about their 2016 paper. We were not thinking of this problem in the context of permutation classes.

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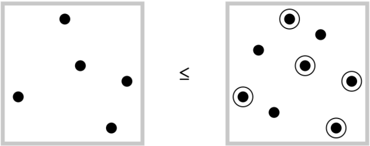
On the effective and automatic enumeration of polynomial permutation classes

Cheyne Homberger [a], Vincent Vatter [b,1]

Permutation classes

The permutation π of length n **contains** the permutation σ of length k if π has a subsequence of length $k \leq n$ that is **order isomorphic** to σ, i.e., that has the same pairwise comparisons as σ.

For example, the subsequence 38514 is order isomorphic to 25413, so 25413 is contained in the permutation 36285714.

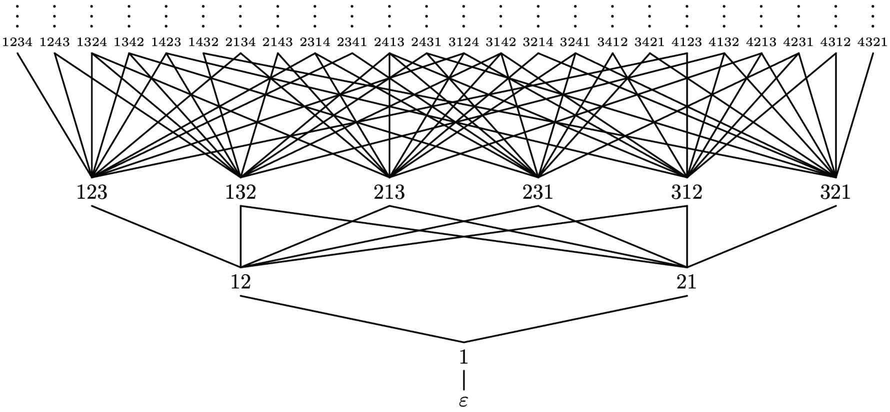


[1]

A **permutation class** $\mathcal{C}$ is a set of permutations (of different lengths) closed under containment. In other words, a permutation class $\mathcal{C}$ is a downset of permutations in this ordering: If $\mathcal{C}$ is a permutation class and $\pi \in \mathcal{C}, \sigma \leq \pi$, then $\sigma \in \mathcal{C}$.

[1]Credit: Vince Vatter

Permutation classes



Credit: Vince Vatter.

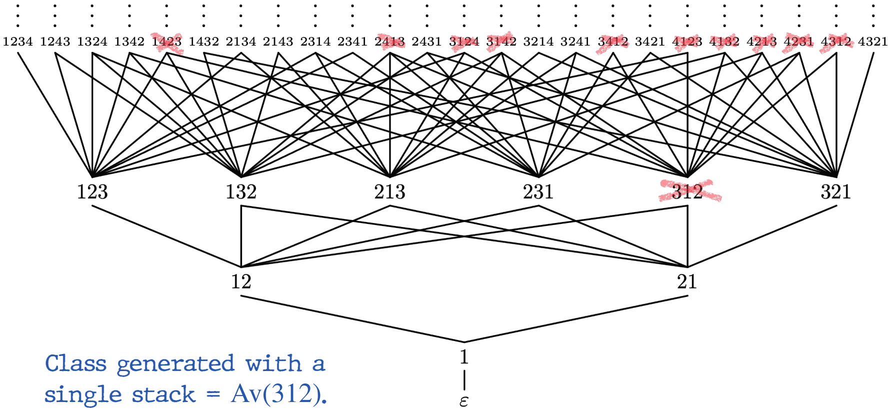
Basis of a permutation class

Given any set of permutations B define

$$\text{Av}(B) = \{\pi : \pi \text{ ``avoids' } \pi \in B\}$$

Given any permutation class $\mathcal{C}$, there is a set B so that $\mathcal{C}$ =Av(B). In fact, we can take B to be the set of minimal permutations not in $\mathcal{C}$, and that is called the **basis** of $\mathcal{C}$.

Class Av(312)



Class generated with a single stack = Av(312).

Credit: Vince Vatter.

Enumerating permutation classes

Let $\mathcal{C}_n := \mathcal{C} \cap S_n$ (S_n denoting the symmetric group)

What is the behavior of the sequence $|\mathcal{C}_0|$, $|\mathcal{C}_1|$,...

One way of doing this is to compute explicitly the generating function of the class,

$$\sum_{n\geq 0} |\mathcal{C}_n| x^n$$

Is it rational? algebraic? D-finite?

One of the first results in the subject is that if $\mathcal{C} = \mathrm{Av}(123)$, then $|\mathcal{C}_n| = \binom{2n}{n}/(n+1)$ (the Catalan numbers).

Stanley-Wilf Conjecture (Theorem since 2004)

Stanley and Wilf independently conjectured the following.

Theorem (Marcos and Tardos, 2004)

For every permutation β, there is a constant C such that for all n,

$$|Av(\beta) \cap S_n| \leq C^n.$$

In other words, given any **proper** permutation class $\mathcal{C}$,

$$\limsup_{n \to \infty} \sqrt[n]{|\mathcal{C}_n|} \text{ is finite.}$$

This limit is called **the (exponential) rate of growth** of $\mathcal{C}$.

Permutation classes enumerated by polynomials

The *Fibonacci Dichotomy* of Kaiser and Klazar (2003) : If $\mathcal{C}$ is a permutation class with $|\mathcal{C}_N| < F_N$ for **some** integer N, then $|\mathcal{C}_n|$ is given by a polynomial for *sufficiently large* n (or, **eventually polynomial**). In other words, there exists n_0 such that $|\mathcal{C}_n| = p(n)$ for all $n > n_0$ where $p(n)$ is a polynomial. Here F_n denotes the nth Fibonacci number.

Homberger and Vatter (2016) provided an algorithm, **HVA**, that finds this polynomial.

It turns out that the permutations that take *at most* k flips to be sorted is a permutation class which can be *eventually* enumerated by polynomials (it might miss the first few values of n.)

Enter generalized permutations $S(m,n) := C_m \wr S_n$

Consider the group $S(m,n) := C_m \wr S_n$, where C_m is the cyclic group of order m, say with elements $\{0,\dots,m-1\}$. We take the following notation: Every element in $S(m,n)$ has the form

$$\pi_1^{e_1}\pi_2^{e_2}\cdots\pi_n^{e_n}, \text{ where } e_i \in C_m.$$

We refer to $\pi_1,\dots,\pi_n$ as the **symbols** and $e_1,\dots,e_n$ as the **signs**.

For example, $3^0 2^1 4^3 5^2 1^0 6^1 \in S(4,6)$.

Notice $S(1,n) \cong S_n$ and $S(2,n) = B_n$ (Coxeter group of type B, the hyperoctahedral group, the group of symmetries of the hypercube.)

Classes in $S(m, n)$

Definition

Given $\pi = \pi_1^{a_1} \cdots \pi_{n_1}^{a_{n_1}} \in S(m, n_1)$ and $\sigma = \sigma_1^{b_1} \cdots \sigma_{n_2}^{b_{n_2}} \in S(m, n_2)$, with $n_1 \leq n_2$, we say that π **contains** σ, denoted by $\pi \leq \sigma$, if there exists $\sigma_{i_1}^{b_{i_1}} \cdots \sigma_{i_{n_1}}^{b_{i_{n_1}}}$ such that $b_{i_j} = a_j$ and the characters in $\sigma_{i_1} \cdots \sigma_{i_{n_1}}$ are in the same relative order as the characters of π.

A **generalized permutation class** is set of generalized permutations that is closed under containment.

Example

The following is a finite class (the set of all elements $\leq$ than $2^0 1^1 3^1 \in S(2,3)$)

$$2^0 1^1 3^1, 2^0 1^1, 1^0 2^1, 1^1 2^1, 1^0, 1^1, \varepsilon$$

$2^0 1^1 \not{3}^{\not{1}}$ gives $2^0 1^1$, $2^0 \not{1}^{\not{1}} 3^1$ gives $1^0 2^1$, etc.

The case $m = 2$

The case $m = 2$ is interesting because it has to do with genome rearrangements

Relations to genome rearrangements

Transforming Cabbage into Turnip: Polynomial Algorithm for Sorting Signed Permutations by Reversals

SRIDHAR HANNENHALLI

Bioinformatics, SmithKline Beecham Pharmaceuticals, King of Prussia, Pennsylvania

AND

PAVEL A. PEVZNER

University of Southern California, Los Angeles, California

Journal of the ACM, 1999

Genome Rearrangement

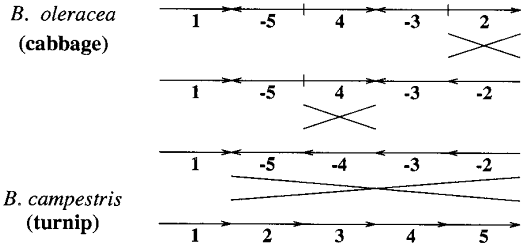


FIG. 1. "Transformation" of cabbage into turnip. Mitochondrial DNA of cabbage and turnip are composed of five conserved blocks of genes that are shuffled in cabbage as compared to turnip. Every conserved block has a direction that is shown by + or − sign.

Journal of the ACM, 1999

Burnt pancake graph BP_n

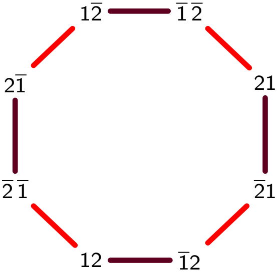
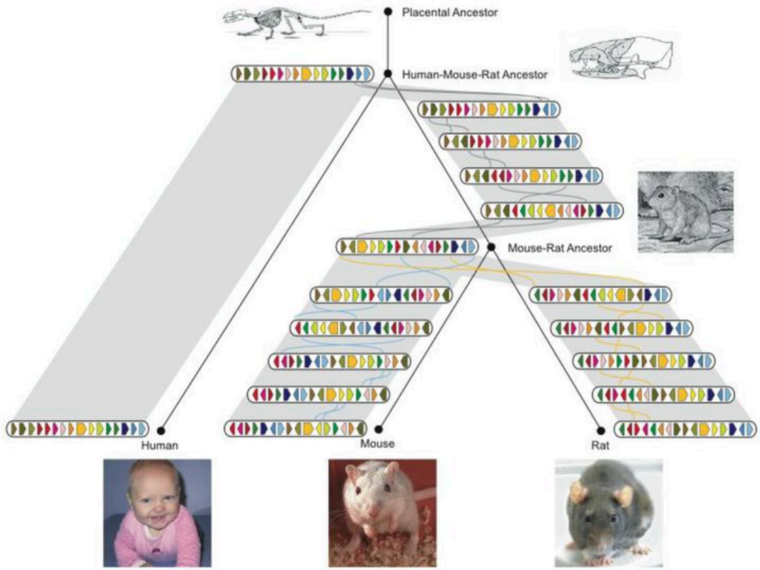


Figure: BP_2 is an 8-cycle.



Rat Consortium, *Nature*, 2004

How many stacks of n burnt pancakes take exactly k flips to be sorted?

A great deal is known about the structure of cycles in BP_n. In particular, we know that 8-cycles are the shortest possible cycles.

1. If $k = 0$, then 1.
2. If $k = 1$, then n.
3. If $k = 2$, then $n(n-1)$.
4. If $k = 3$, then $n(n-1)^2$.

How many stacks of n burnt pancakes take exactly 4 flips to be sorted?

Theorem

(B., Buehrle and, Patidar 2019) If $n \geq 1$, there are

$$\frac{1}{2}n(n-1)^2(2n-3)$$

stacks of n burnt pancakes that take exactly 4 flips to be sorted.

The proof used elementary methods like the our classification of and 8 and 9-cylces in BP_n and PIE. It was a lot of book keeping.

There's gotta be a better way!

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There's gotta be a better way! There is.

Generalized peg permutations

Generalized peg permutations are generalized permutations in which each entry is also associated with a decorator in the set $+, -, \bullet$. For example, $3^{0\bullet}1^{1+}2^{0+}$.

Containment is defined similarly, with:

- $+$ contains $+$ and $\bullet$,
- $-$ contains $-$ and $\bullet$, and
- $\bullet$ only contains $\bullet$

Inflating a signed permutation

For a peg permutation π and a vector $\mathbf{v}$ each of length n, we say that the *inflation* of π by $\mathbf{v}$, written $\pi[\mathbf{v}]$, is the generalized permutation obtained by replacing each entry in π with an interval according to the following rules:

- $\pi(i)$ is replaced with an interval of length $\mathbf{v}(i)$.
- If $\pi(i)$ is decorated with a $+$, its interval must be increasing, and if $\pi(i)$ is decorated with a $-$, its interval must be decreasing.
- If $\pi(i)$ is decorated with a $\bullet$, then $\mathbf{v}(i)$ must be 0 or 1.
- Every entry in the interval corresponding to $\pi(i)$ has the same sign as $\pi(i)$.
- The relative order of the intervals matches the relative order of the entries of π.

Example

If $\pi = 1^{1+}3^{0\bullet}2^{0-}$ and $\mathbf{v} = \langle 4, 1, 3 \rangle$. Then $\pi[\mathbf{v}] = 1^1 2^1 3^1 4^1 \quad 8^0 \quad 7^0 6^0 5^0$.

Grid generalized permutation classes

The grid class of a **peg** permutation π, of length k, written $\mathrm{Grid}(\pi)$, is the set of generalized permutations

$$\{\pi[\mathbf{v}] : \mathbf{v} \in (\mathbb{Z}_{\geq 0})^k\}$$

If Π is a set of peg permutations (not necessarily of the same length) then $\mathrm{Grid}(\Pi)$ is given by

$$\{\mathrm{Grid}(\pi) : \pi \in \Pi\}.$$

It is easy to see that $\mathrm{Grid}(\Pi)$ is closed under containment (thus a permutation class).

Fibonacci Dichotomy extends to generalized permutation classes

This is our **main theorem**.

Theorem (B. and Skora, 2024)

Let $\mathcal{C}$ be a signed permutation class with m signs. Then the following are equivalent

1. $|\mathcal{C} \cap S(m,n)| < F_n$ *for some n (Fibonacci numbers)*
2. *There exists n_0 such that $|\mathcal{C} \cap S(m,n)|$ is polynomial for all $n > n_0$.*
3. *$\mathcal{C}$ is a grid generalized permutation class.*

Essentially, we have **characterized** all generalized permutation classes that are enumerated by polynomials.

Huczynska and Vatter (2006) and Albert et al. (2013) established the theorem for $m = 1$. Homberger and Vatter (2016) provided an algorithm (HVA) that produces these polynomials. The techniques extend to $m > 1$.

Idea of the proof

If $\mathbf{v} \in (\mathbb{Z}_{\geq 0})^n$, and $\sum \mathbf{v}(i)$ be the **weight** of **v**. Then the **downset** of **v**

$$\{\mathbf{w} \in (\mathbb{Z}_{\geq 0})^n \mid \mathbf{w}(i) \leq \mathbf{v}(i)\}.$$

Theorem

Let C denote a downset in $(\mathbb{Z}_{\geq 0})^n$. For sufficiently large n, the number of vectors in C of weight n is given by a polynomial.

Stanley proposed the theorem as a 1976 *Monthly* problem offering two solutions. One of these solutions is “elementary“”’ and the second one considers the downset as Hilbert functions.

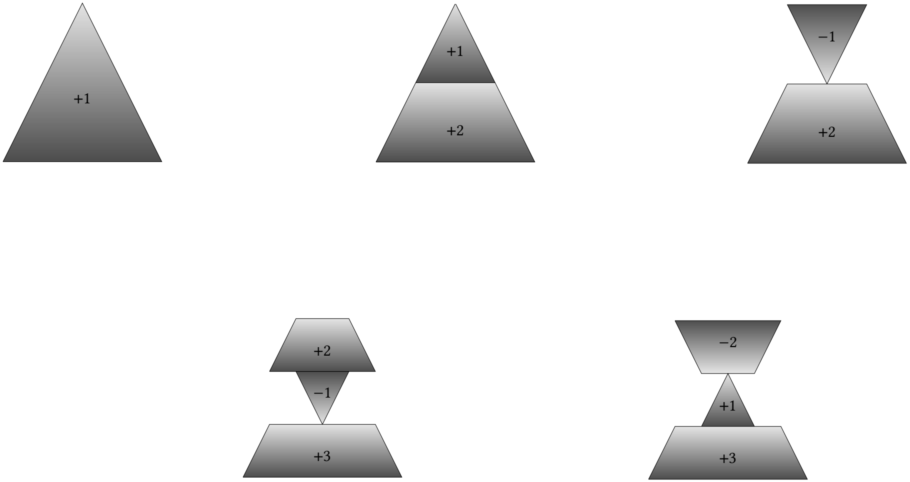
It turns out that the elements of grid class correspond to downsets.

Enumerating grid generalized permutation classes

Homberger and Vatter (2016) gave an algorithm to enumerate permutation classes that are eventually polynomial.

We have extended the algorithm to work for generalized permutation classes

Code available
https://github.com/danskora/SignedPermutationClasses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Stacks of burnt pancakes that take at most k flips to be sorted

- $\mathrm{Grid}(\{1^1 2^0\})$ is the class representing the stacks of burnt pancakes that take at most one flip to be sorted. The algorithm outputs $n+1$.
- $\mathrm{Grid}(\{2^0 1^1 3^0, 2^1 13\})$ is the class representing the stacks of burnt pancakes that take at most two flips to be sorted. The algorithm outputs n^2+1.
- There are

$$n^5 - \frac{29}{6}n^4 + \frac{17}{2}n^3 - \frac{26}{6}n^2 + \frac{1}{2}n + 1$$

  stacks of burnt pancakes that require at most 5 flips to be sorted, etc...
- These polynomials work for all $n \geq 1$ (they don't miss any values like in the case of S_n.)

Pancakes with more than 2 sides?

Back to $S(m, n) = C_m \wr S_n$.

Recall the notation: $4^2 2^0 1^2 3^1 \in C_3 \wr S_4$. The characters from S_n are the **symbols** and the characters from C_m are the signs.

Given $\pi \in S(m, n)$. Then the action of r_i to π reverses the first i symbols in π and adds $1 \mod m$ to the affected signs.

$$r_3(4^2 2^0 1^2 3^1) = 1^0 2^1 4^0 3^1$$

and

$$r_4(4^2 2^0 1^2 3^1) = 3^2 1^0 2^1 4^0$$

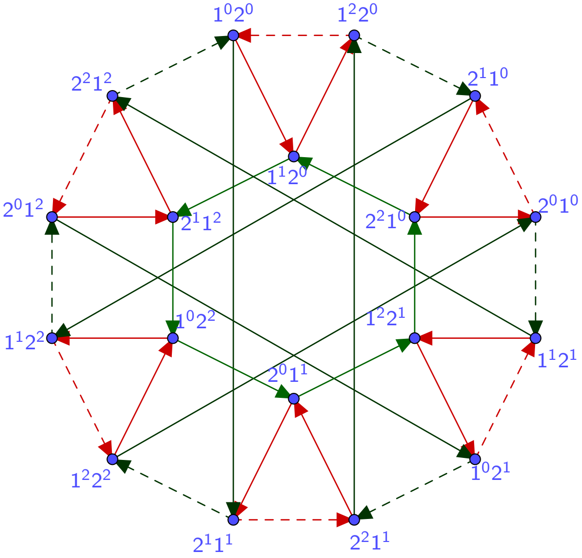
Generalized pancake graphs

These are the *Cayley graphs* of $C_m \wr S_n$ using $\{r_1(e), \ldots, r_n(e)\}$ as generators ($e \in C_m \wr S_n$ is the identity element).

If $m = 1$ we recover the pancake graph P_n. If $m = 2$ we recover the burnt pancake graph BP_n. Moreover, if $m = 1, 2$ the r_is are **involutions**

If $m \geq 3$, we obtain a directed graph.

$C_3 \wr S_2$


1⁰2⁰
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2²1²
2¹1⁰
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2⁰1²
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1⁰2²
1²2¹
1¹2²
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2⁰1¹
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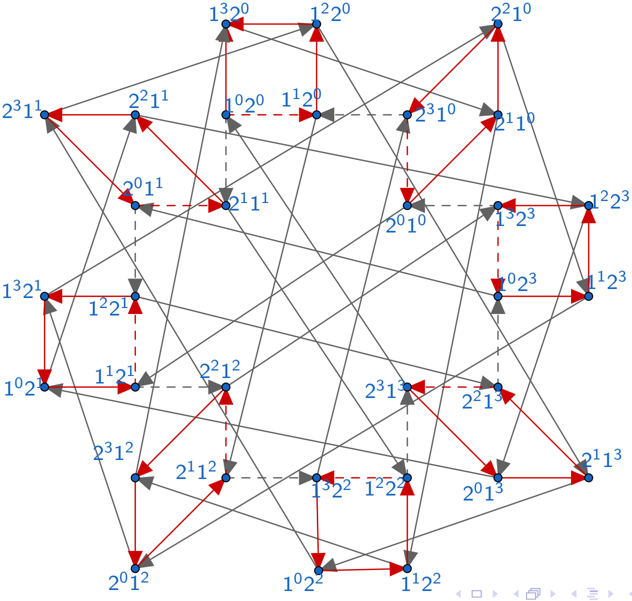
1⁰2¹
2¹1¹
2²1¹

$C_4 \wr S_2$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Application of our main theorem

Corollary

The number of stacks of generalized pancakes that require k flips to be sorted is eventually polynomial.

Summary

1. We talked about the pancake problem
2. We started by counting the number of stacks of pancakes that require 4 flips to be sorted as a motivation
3. We talked about permutation classes and some enumerative results, including the classes that are eventually enumerated by polynomials
4. We defined signed permutation classes. These objects are very natural, but they haven't been studied as much.
5. We characterize the signed permutation classes that are eventually enumerated by polynomials. This was our main result.
6. We applied our result to show that the number of generalized pancake stacks that require k flips to be sorted is eventually polynomial.
7. Future directions: explore other enumerative questions relating to signed permutation classes.

Thank you!