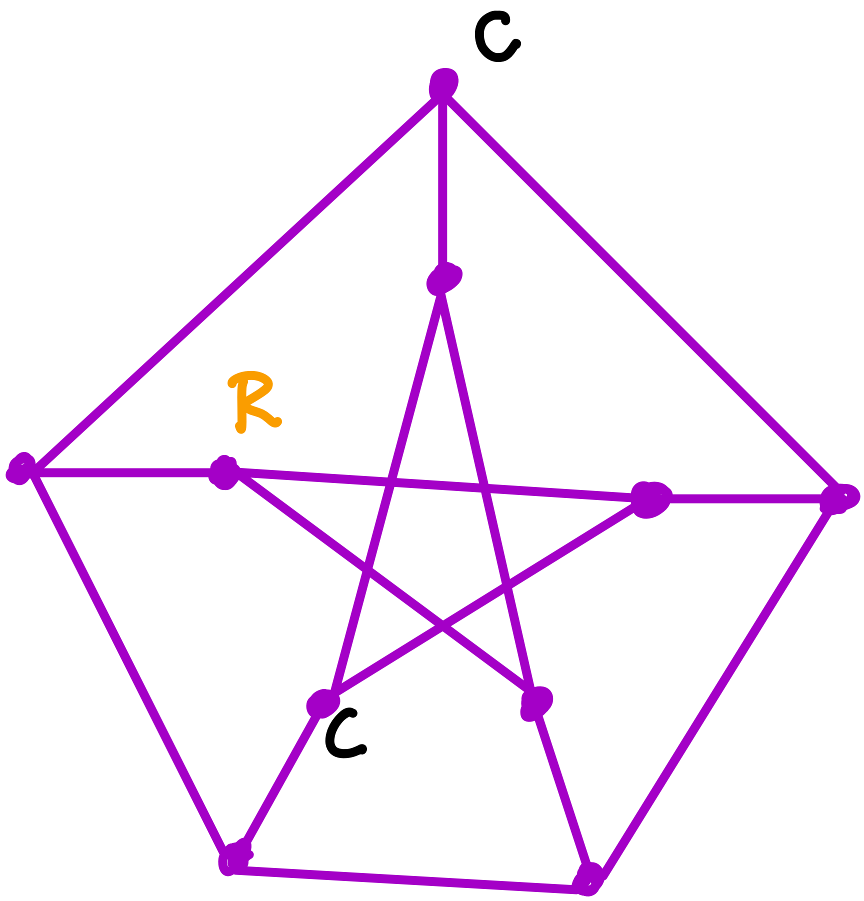
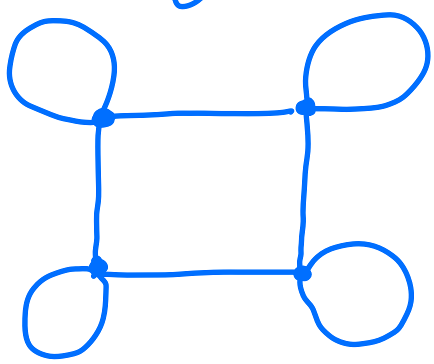
Recent Advances in Pursuit-Evasion Games on Graphs



Cops and Robbers

G, reflexive graph



Cop Player
moves cops

Robber player
moves a robber

Round Zero: Each cop is assigned to a vertex. Then the robber is placed.

Round One: Cops move — each moves along one edge. Then Robber moves

Continue ...

Capture (same vertex) is the win condition for Cops

Perpetually avoiding capture is the win condition for ROBBER.

How many pawns?

If cops can win every play on G with k cops, say G is k-cop win.

Cop number, $c(G)$, is the least k such that G is k-cop win.

- complete information
- discrete time steps / unit speed
- pawns can "pass"
- cops can move independently

- two-way streets
- win of capture

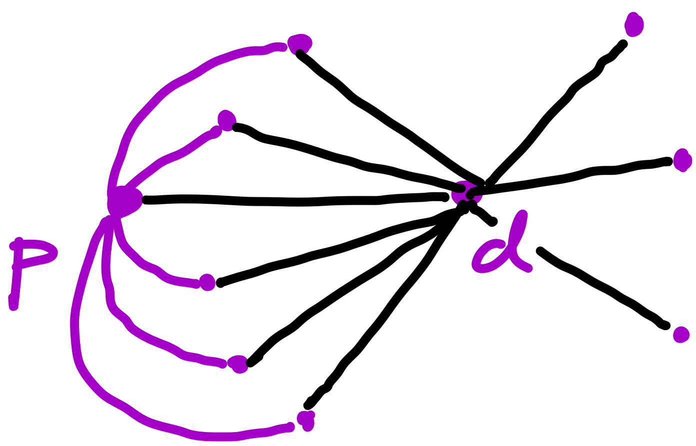
Variations:

change or relax one of the above.

- lazy cops
- tandem cops
- continuous movement
- directed graphs
- capture at a distance

Is this mathematically interesting / Rich?

$c(G) = 1 \iff G$ dismantlable for G finite



Say d dominates the pitfall p.

$N[p] \subset N[d]$

$N(p) = \{ v \in V(G) \mid pv \in E(G) \}$
$N[p] = N(p) \cup \{p\}$

Theorem: If p is a pitfall, then $c(G) = c(G-p)$.

Due to Quilliot, Nowakowski & Winkler.

Idea of copnumber due to Aigner & Fromme.

Open Problem: characterize digraphs with cop number 1.

Theorem: G planar, connected, finite $\Longrightarrow$ $c(G) \le 3$

Due to Quilliot, Aigner & Fromme

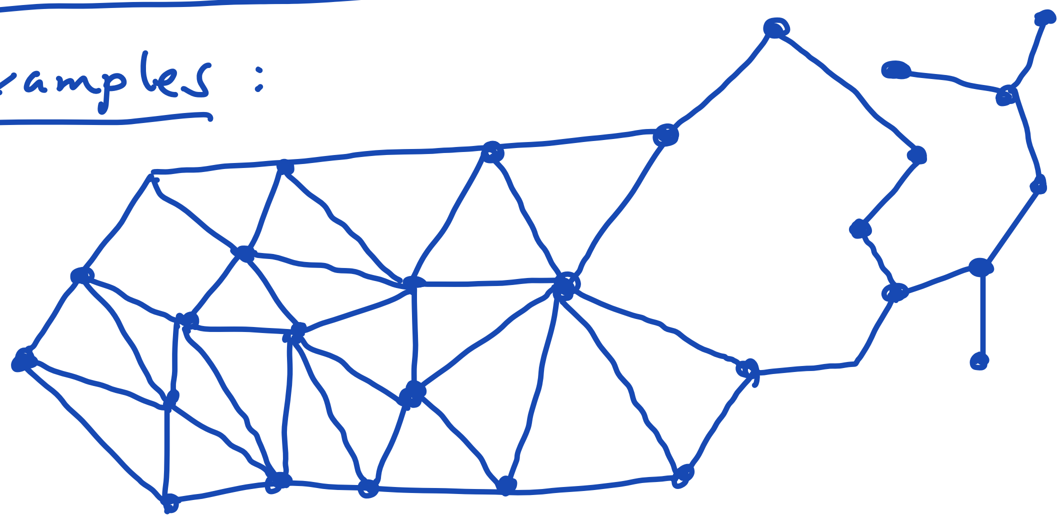
Open Problem: If G genus g, finite, connected, is $c(G) \le g+3$?

Theorem: If $\delta(G) = d+1$, minimum degree, and $\text{girth}(G) \geq 8k-3$, then $c(G) > d^k$

Due to Franky!

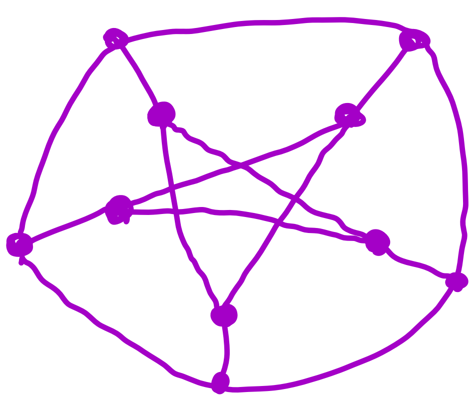
Examples:

(1)



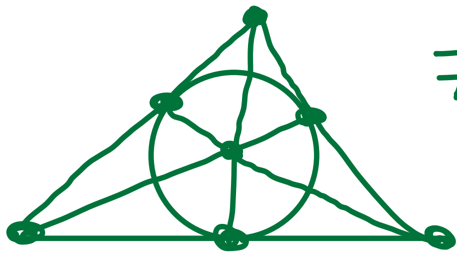
$c(G) =$ __________

(2)



$c(G) =$ __________

(3) P, finite projective plane



7 points

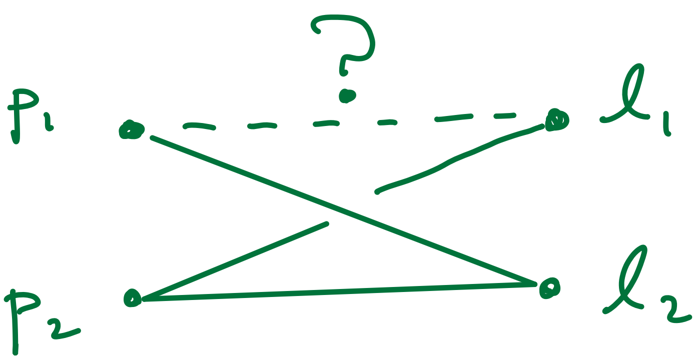
7 lines

$G = I$ (incidence graph)

bipartite $\{\text{points}\} \cup \{\text{lines}\} = V(G)$

edges $\longleftrightarrow$ incidence.

$c(G) =$ __________



two points determine a line

projective plane of order K

has $|I| = 2(K^2+K+1)$

and $c(G) = K+1$

So, $c(G) \in O(\sqrt{|G|})$

Is $c(G) \in O(\sqrt{|G|})$ for every finite connected graph?

Open Problem! Meyniel's Conjecture.

METHODS

Shadow strategy

A subgraph H of G is a retract if $\exists f: G \to H$ graph hom s.t.

$f \circ i_H = id_H, \quad i_H : H \hookrightarrow G$ inclusion.

Lemma If H is a retract of G, then $c(H) \leq c(G)$

* where you start in a connected graph does NOT matter

If k cops win on G by starting on $v_1, \dots, v_k$, then start on $(f(v_i))$

If the winning strategy says to play $(v_i) \dashrightarrow (v_i')$, then play $(f(v_i'))$.

Corollary: $c(G-p) = c(G)$ if p is a pitfall.

$f : G \longrightarrow G-p$
$f(p) = d$, dominating vertex

$c(G-p) \le c(G)$ by Lemma.

For the reverse direction, if k cops win on $G-p$, then they can play and capture the Robber's shadow, $f(R)$. When they do, either $f(R) = R$ or $f(R) = d$ and R is at p.

Corollary: G finite and dismantlable $\Longrightarrow$ $c(G) = 1$.

Guarding subgraphs

H is k-guardable if after finitely many rounds, k cops move only in H and if R were to move to H, then he is captured on the next cop move.

<u>Lemma</u>: If H is a retract of G, G is connected, and $c(H) \le k$, then H is k-guardable.

Since G is connected, k cops can move to H and thereafter move only in H. Cop capture the robber's shadow $f(R)$, where $f: G \to H$ is a retraction.

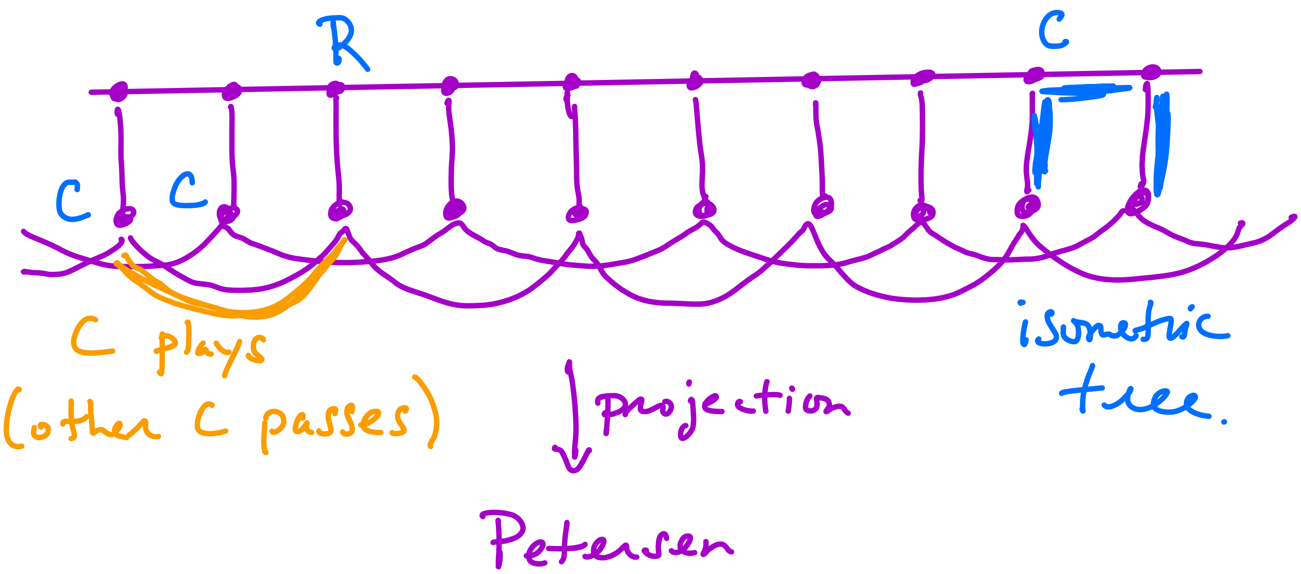
Once $f(R)$ is captured, continue to capture: if R moves from v to v', C moves from $f(v)$ to $f(v')$. If R were to move to H, then $f(v') = v'$.

Note: Only one cop is needed once the robber's shadow is captured.

Theorem (Nowakowski & Rival) If T is an isometric subtree of a reflexive graph G, then T is a retract of G.

Corollary: Finite isometric trees are 1-guardable.

Another look at the Petersen graph



$GP(n,2)$ vertices: $\mathbb{Z}_n \times \mathbb{Z}_2 = \{a_i\} \cup \{b_i\}$

edges: $a_i a_{i+1}$ $a_i b_i$

$b_i b_{i+2}$

$GP(\infty, 2)$ vertices $\mathbb{Z} \times \mathbb{Z}_2 = \{a_i\} \cup \{b_i\}$

projection is is the graph hom induced by $\mathbb{Z} \longrightarrow \mathbb{Z}_n$

$k \longmapsto [k]$

Theorem: $c(GP(n,k)) \leq 3$ if $k = 2, 3$; $c(GP(n,k)) \leq 4$ for $1 < k \leq n/2$.

joint work with Ball, Guzman, Niemi-Colvin, Schonsheck

Weak Cops and Robbers

Win condition for cops is changed to having a strategy such that the robber cannot visit any vertex infinitely often

(definition due to Lehner)

* Game ends if cops capture robber

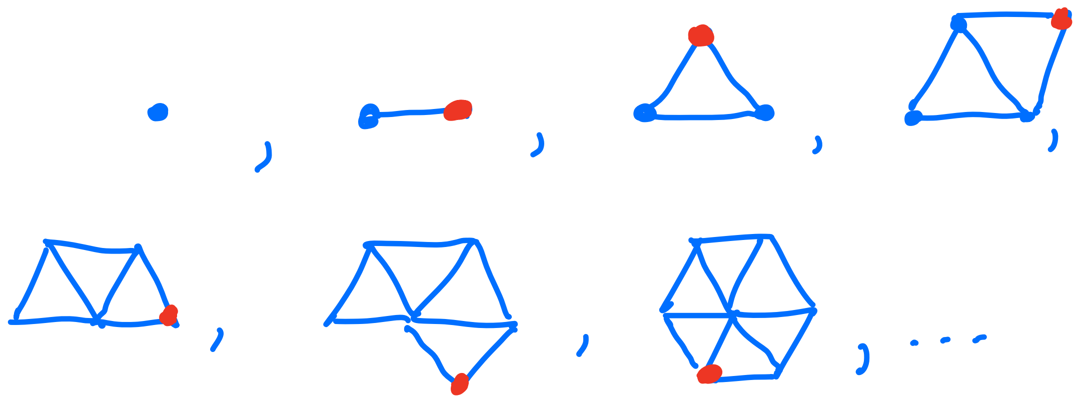
$wc(G)$, weak cop number.

Examples: ① $wc(G) = c(G)$ if G is finite.

② $wc(T) = 1$ if T is a tree.

③ $wc(G) = 1$ if G is the underlying graph of the triangulation of Euclidean plane by unit equilateral triangles.

G is constructible if $G = \bigcup_{i \geq 0} G_i$ such that $V(G_i) = \{0, \ldots, i\}$ and i is a pitfall in G_i with some dominating vertex in G_{i-1}, $i \geq 1$.



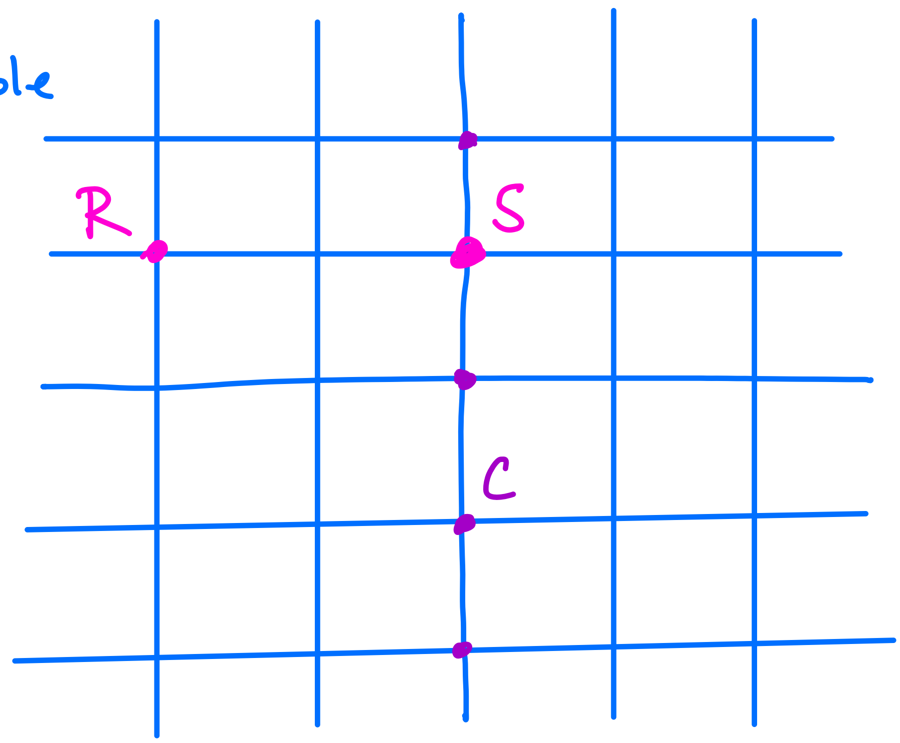
Theorem (Lehner) If G is constructible, then $wc(G) = 1$.

Open problem: characterize (locally finite) graphs that are weakly cop win.

Theorem: If G is connected, locally finite, and admits a planar embedding with no vertex accumulation points, then $wc(G) \leq 3$.

Methods: Isometric trees are weakly guardable: after finitely many rounds, cops only play in T and either capture the robber's shadow or, in avoiding his shadow's capture, the robber will lose the weak game.

Example



This is joint work with SURIEM REU students: Dubeau & Matys and Diep-Nguyen & Minor

<u>Lemma</u>: If G is a locally finite, connected graph, then $wc(G) \le k$ $\Longleftrightarrow$ k cops can win every play by capturing or playing so that

$$\lim_{r \to \infty} d(x_0, v_r) = \infty, \text{ where}$$

v_r is the position of the robber on round r, x_0 is a base point, and d is the length of a shortest path.

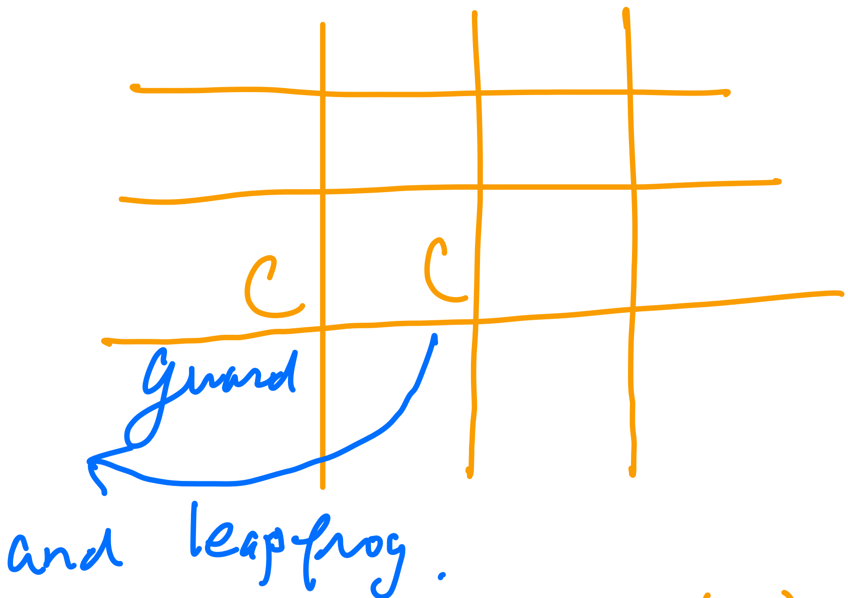
Simple idea: If R must leave every r-ball centered at x_0, then R cannot visit any vertex infinitely often.

Conversely, if R visits some r-ball centered at x_0 infinitely often, then R visits some vertex infinitely often because the r-ball has finitely many vertices (locally finite graph)

So, to weakly guard an isometric tree, one cop plays in the tree and captures the shadow or, failing this, $d_T(x_0, v_r) \to \infty$

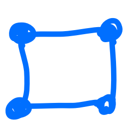
but $d_T(x_0, v_r) = d_G(x_0, r)$
because the tree T is isometric.

<u>Application</u>: The square grid G
has $wc(G) = 2$.



H retract of G

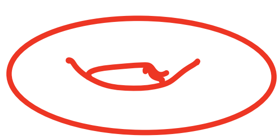
$$wc(G) \le wc(H)$$
$$\le c(H)$$

Choose $H =$  $\therefore\ wc(G) \le 2$.

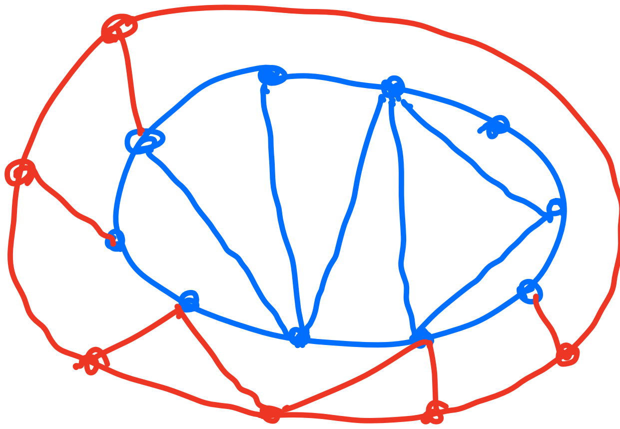
Recent Directions / Results.

• Ivan & Leader & Walters construct a locally constructible graph that is not weak cop win

• Morris & Runte & Skelton show $c(GP(n,k)) = 4$ if it has girth 8. Also if G finite, connected has min degree ≥ 3 and girth ≥ 9, then $c(G) > \delta$

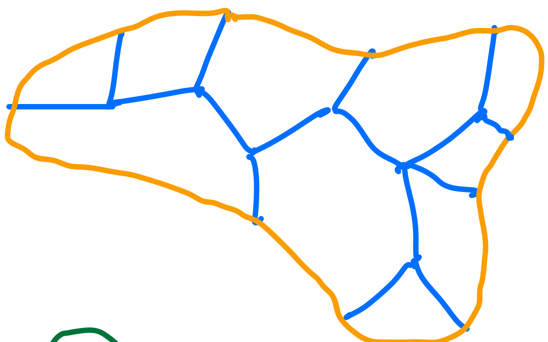
• Lehner proves $c(G) \leq 3 + 0$ if G is finite, connected, genus 1; i.e. embeds in  .

Families of graphs



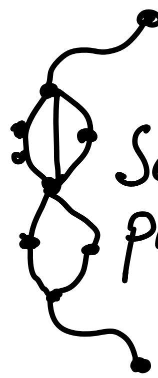
outerplanar

2-outerplanar



Halin

Cactus.



Series parallel

Work with SURIEM REU students George & Knobloch & McMullen & Stewart.

- tandem win
- algorithms
- other variants

<u>Theorem</u> (Joret & Kaminski) For connected finite graphs,

$$c(G) \leq tw(G)/2 + 1,$$ where

$tw(G)$ = tree width.

<u>Theorem</u> (Bodlaender) Halin graphs have treewidth 3.

Outerplanar graphs, cactus graphs have tree width 2

So all have $c(G) \leq \frac{3}{2} + 1$

$\Rightarrow c(G) \leq 2$

k-outerplanar; $tw \le 3k-1$.

2-outerplanar: $tw \le 5$

$$c \le \frac{5}{2} + 1 \Rightarrow c \le 3$$

Do 2-outerplanar graphs have $c \le 2$?