Simplicial Complexes from Graphs

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Simplicial complexes

Definition

A *simplicial complex* on a set V is a nonempty collection of subsets of V closed under taking subsets.

What collections of edges or vertices of a graph form a simplicial complex?

Examples

Let $G = (V, E)$ be a simple graph.

- The subsets $\sigma \subseteq E$ for which the spanning subgraph induced by σ is disconnected forms a simplicial complex.
- The subsets $\sigma \subseteq E$ for which the spanning subgraph induced by σ is connected does not form a simplicial complex in general.
- The independent sets $\sigma \subseteq V$ form a simplicial complex.
- The independent sets $\sigma \subseteq E$ (the matchings) form a simplicial complex.

- Of interest: The topology of the simplicial complexes coming from graphs: homology, connectivity, homotopy type, shellability.
- Dissertation of Jacob Jonsson (2005; published as Springer Lecture Notes in 2008): 18 different simplicial complexes from graphs.

Matching complexes

Lots of work on matching complexes of:

- complete graphs (many people, see Wachs survey)
- complete bipartite graphs, called chessboard complexes (many people, see Wachs survey)
- paths and cycles (Kozlov)
- forests (Marietti and Testa): matching complex of every forest is contractible or homotopy equivalent to a wedge of spheres.
- grid graphs (Braun and Hough, Matsushita)
- honeycombs and caterpillar graphs (Jelić Milutinović, Jenne, McDonough, Vega)
- polygonal line tilings (Jelić Milutinović, et al.; Matsushita)

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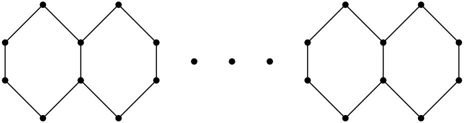
The graphs whose matching complexes are combinatorial manifolds have been characterized (Bayer, Goeckner, Jelić Milutinović)

Background

Jelić Milutinović, et al., got connectivity bounds for the matching complexes of strings of $2n$-gons for a fixed n.
Matsushita then found the homotopy types of these graphs.



Joint work with Marija Jelić Milutinović and Julianne Vega, thanks to MSRI Summer Research in Mathematics Program.

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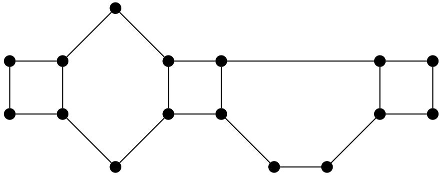
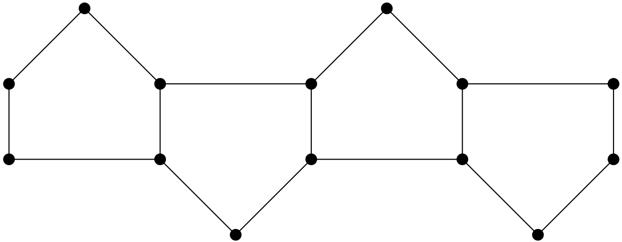
More general "polygonal line tilings": For a sequence $s_1, s_2, \ldots, s_n$ of integers, each ≥ 4, let $G_{s_1,s_2,\ldots,s_n}$ be a graph formed by connecting s_i-gons in a line, where no three share a vertex.



Theorem

The matching complex of $G_{s_1,s_2,\ldots,s_n}$ is contractible or homotopy equivalent to a wedge of spheres.

In the case where all the polygons are pentagons, we have an exact formula for the homotopy type and it involves Fibonacci numbers!



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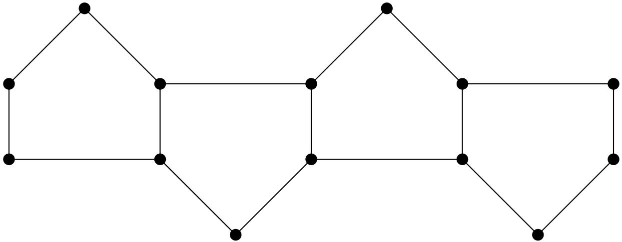


Theorem

Let G be the pentagonal line tiling with t pentagons. Then the matching complex of G satisfies $\mathcal{M}(G) \simeq \bigvee_{F_{t+2}-1} \mathbb{S}^t$.

Methods

We use theorems adapted (stolen) from theorems about independence complexes. Here $EN[e]$ is the closed edge neighborhood and $EN(e)$ is the open edge neighborhood of an edge e.

Theorem (Engström)

Let G be a graph that contains two different edges e and h such that $EN(e) \subset EN(h)$. Then $\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{h\})$.

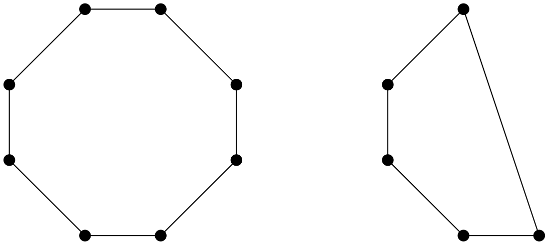
Theorem (Adamaszek)

Let G be a graph that contains two different edges e and h such that $EN[e] \subset EN[h]$. Then

$$\mathcal{M}(G) \simeq \mathcal{M}(G \setminus \{h\}) \vee \Sigma\mathcal{M}(G \setminus EN[h]).$$

Theorem (Engström)

If G is a graph with a path X of length 4 whose internal vertices are of degree two and whose end vertices are distinct, then $\mathcal{M}(G) \simeq \Sigma\mathcal{M}(G/X)$, where G/X is the contraction of X to a single edge with endpoints given by the endpoints of X.



Other results

Lines of triangles. They don't fit into the definition of polygonal line tilings, because three triangles intersect at a vertex. But we also found the homotopy type of lines of triangles, using the same methods.

Perfect matching complex: If G has a perfect matching, the subcomplex of the matching complex with facets corresponding to perfect matchings. We have results on the perfect matching complexes of honeycomb graphs. Here we use the connection with plane partitions, and homotopy type is computed using discrete Morse theory.

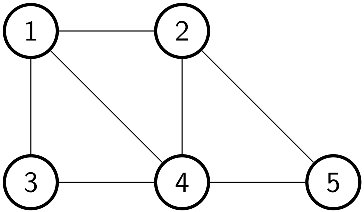
k-Cut Complexes

Definition

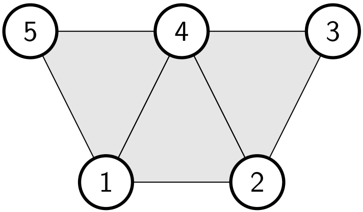
Let $G = (V, E)$ be a simple graph, and let $k \geq 2$.
Let $D_k(G) = \{S \subseteq V \mid G[S] \text{ is disconnected and } |S| = k\}$.
The *k-cut complex* $\Delta_k(G)$ is the simplicial complex whose facets are the sets $F \subseteq V$ such that $V \setminus F \in D_k(G)$.

σ is a face of $\Delta_k(G)$ if and only if its complement $V \setminus \sigma$ contains a subset S of size k such that the induced subgraph $G[S]$ is disconnected.

Graph G



$\Delta_2(G)$



$\Delta_2(G) = \langle 145, 124, 234 \rangle$

Joint work with Mark Denker, Marija Jelić Milutinović, Rowan Rowlands, Sheila Sundaram and Lei Xue. A GRWC project.

Motivation: Generalize Fröberg's theorem on linear resolutions of edge ideals of graphs, reinterpreted by Eagon and Reiner:

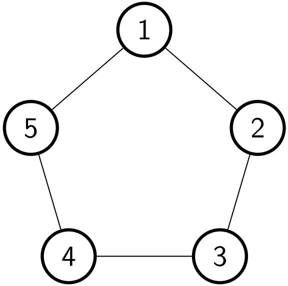
Theorem

The following are equivalent:

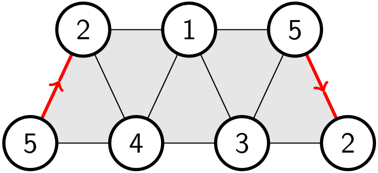
- *G is chordal*
- *$\Delta_2(G)$ is Cohen-Macaulay*
- *$\Delta_2(G)$ is shellable*
- *$\Delta_2(G)$ is vertex decomposable.*

A Non-Chordal Graph

The Graph C_5



$\Delta_2(C_5)$



$\Delta_2(C_5) = \langle 245, 124, 134, 135, 235 \rangle$

The 2-cut complex for C_5 is a Möbius band, which is not shellable.

Theorem (Behague)

Every pure simplicial complex is the k-cut complex of some chordal graph for some k.

We consider specific classes of graphs. For G in a class of graphs:

- Is $\Delta_k(G)$ homotopy equivalent to a wedge of spheres?
- If so, how many spheres, and of what dimension?
- Is $\Delta_k(G)$ shellable?

Shellability

Definition

A *shelling* of a (pure) simplicial complex Δ is an ordering $F_1, F_2, \ldots, F_t$ of the facets of Δ such that for every $1 < j \leq t$,

$$\left(\bigcup_{i=1}^{j-1} F_i\right) \bigcap F_j$$

is a simplicial complex all of whose facets have cardinality $|F_j| - 1$. If Δ has a shelling, Δ is called *shellable*.

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If a simplicial complex is shellable, then it is contractible or homotopy equivalent to a wedge of spheres.

Examples of shellability results

- For a graph G, $\Delta_k(G)$ is shellable if and only if $\Delta_k(H)$ is shellable for all induced subgraphs of G.
- If G is chordal, then $\Delta_2(G)$ and $\Delta_3(G)$ are shellable.
- For $k \geq 4$, there are chordal graphs with $k+2$ vertices for which $\Delta_k(G)$ is not shellable.
- If G is a tree, then $\Delta_k(G)$ is shellable for all $k \geq 2$.
- If G is a threshold graph, then $\Delta_k(G)$ is shellable for all $k \geq 2$.

Examples of homotopy results

Methods include discrete Morse theory.

- Let C_n be the cycle graph, $n \geq k+2 \geq 5$. Then

$$\Delta_k(C_n) \simeq \bigvee_{\binom{n-1}{k-1}-n} \mathbb{S}^{n-k-1}.$$

- Let $K_{m,n}$ be the complete bipartite graph.
    - If $m < k \leq n$, then $\Delta_k(K_{m,n})$ is contractible.
    - If $m \leq n < k$, then $\Delta_k(K_{m,n})$ is the void complex.
    - If $k \leq m \leq n$, then $\Delta_k(K_{m,n})$ is homotopy equivalent to a wedge of $\binom{m-1}{k-1}\binom{n-1}{k-1}$ spheres of dimension $m+n-2k$.

Related and ongoing work

- Perfect matching complexes
- Extending matching complex results to more general planar graphs.
- Total cut complexes: facets are complements of independent sets of vertices
- Cut complexes of grid graphs and squared cycles
- f-vectors and h-vectors of cut complexes
- Euler characteristics and Betti numbers of cut complexes

The end

THANK YOU!

http://www.math.ku.edu/~bayer

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