On the h^*-polynomial of type C hypersimplices

Jose Bastidas
LACIM–Université du Québec à Montréal

Joint work with Antoine Abram (LACIM)

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Outline

GOAL:

Theorem (Abram, B.)

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w \in X_n \\ \mathrm{cdes}(w^{-1}) = k}} t^{\mathrm{basc}(w)}$$

Theorem (Abram, B.)

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w \in X_n \\ \mathrm{fexc}(w) = k-1}} t^{\mathrm{des}_w(w)}$$

+ relation with
- Eulerian numbers of type B
- Lattice of strict partitions

I. Eulerian numbers & hypersimplices

II. Ehrhart & h^* polynomials

III. Φ-hypersimplices

IV. Proofs

Eulerian Numbers

$$A_{n,k} := \#\{ w \in \mathfrak{S}_n : \mathrm{des}(w) = k \}$$

$i \in [n-1]$ is a descent of w if $w_i > w_{i+1}$

Coefficients of Eulerian polynomial:

$$A_n(t) := \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)}$$

$$\mathfrak{S}_3 = \left\{ \begin{matrix} & \underline{2}13 & 2\underline{3}1 & \\ 123 & & & \underline{3}\,\underline{2}1 \\ & 1\underline{3}2 & \underline{3}12 & \end{matrix} \right\} \qquad A_3(t) = 1 + 4t + t^2$$

Other Eulerian statistics:
- ascents
- excedances

EULERIAN NUMBERS

$$A_n(t) := \sum_{w \in \mathfrak{S}_n} t^{\operatorname{des}(w)}$$

$$\mathfrak{S}_3 = \left\{ 123 \quad \begin{matrix} \underline{2}13 \\ 1\underline{3}2 \end{matrix} \quad \begin{matrix} 2\underline{3}1 \\ \underline{3}12 \end{matrix} \quad \underline{3}\,\underline{2}1 \right\} \qquad A_3(t) = 1 + 4t + t^2$$

Also: $$A_{n,k} = \operatorname{Vol}\Big(\underbrace{\Delta_{n+1,k+1}}_{\substack{\text{Convex hull of all 0-1 vectors in } \mathbb{R}^{n+1} \\ \text{with exactly } k+1 \text{ ones}}}\Big)$$

Eulerian Numbers

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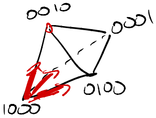
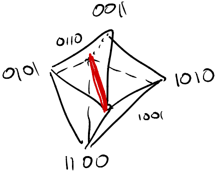
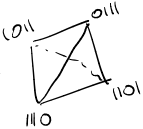
$$\mathfrak{S}_3 = \left\{ 123 \quad \underline{2}13 \quad 1\underline{3}2 \quad 2\underline{3}1 \quad \underline{3}12 \quad \underline{3}\,\underline{2}1 \right\} \qquad A_3(t) = 1 + 4t + t^2$$

Also: $\quad A_{n,k} = \mathrm{Vol}\left(\Delta_{n+1,k+1}\right)$ hypersimplex

normalized volume

$\Delta_{4,1} =$ 

$\Delta_{4,2} =$ 

$\Delta_{4,3} =$ 

$e_2 - e_1, \; e_3 - e_1, \; e_4 - e_1$

Eulerian Numbers

$$A_n(t) := \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)}$$

Recursion:

$$A_{n+1}(t) = (1+nt)\,A_n(t) + t(1-t)\,A_n'(t)$$

Carlitz identity:

$$\sum_{k \geq 0} (k+1)^n\, t^k = \frac{A_n(t)}{(1-t)^{n+1}}$$

h^*-Polynomials

- $\Lambda \subseteq \mathbb{R}^n$ a lattice (think $\Lambda = \mathbb{Z}^n$ or $\Lambda = \frac{1}{2}\mathbb{Z}^n$)
- $P \subseteq \mathbb{R}^n$ a d-dimensional lattice polytope (with vertices in Λ)

Example: $P = [0,1]^n$ $\Lambda = \mathbb{Z}^n$

$P = \Delta_n = \Delta_{n,1}$ standard simplex $\Lambda = \mathbb{Z}^n$

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[Ehrhart 1962] The function $L_P^\Lambda : \mathbb{N} \longrightarrow \mathbb{N} : k \longmapsto \#(kP \cap \Lambda)$

is polynomial of degree d

Example: $P = [0,1]^n$ $\Lambda = \mathbb{Z}^n$ $\Rightarrow$ $L_P^\Lambda(k) = (k+1)^n$

$P = \Delta_n = \Delta_{n,1}$ standard simplex $\Lambda = \mathbb{Z}^n$ $\Rightarrow$ $L_P^\Lambda(k) = \binom{k+n-1}{n-1}$

The leading coefficient of L_P^Λ is the volume of P

h^*-Polynomials

$\Lambda \subseteq \mathbb{R}^n$ lattice

$P \subseteq \mathbb{R}^n$ d-dimensional

$\Rightarrow L_P^\Lambda(k) = \#(kP \cap \Lambda)$ degree d polynomial

The h^*-polynomial of P:

$$\sum_{k \geq 0} L_P^\Lambda(k)\, t^k = \frac{h_P^*(t)}{(1-t)^{d+1}}$$

coefficients of $h_P^*(t)$ express $L_P^\Lambda(k)$

in the basis $\left\{ \binom{k+d}{d}, \binom{k+d-1}{d}, \ldots, \binom{k+1}{d}, \binom{k}{d} \right\}$

$$h_P^*(1) = \text{normalized volume of } P$$

Examples: • $h^*_{[0,1]^n}(t) = A_n(t)$ (Carlitz) $L_{[0,1]}(k) = (k+1)^n$

• $h^*_{\Delta_n}(t) = 1$ $(1-t)^{-n} = \sum_{k\geq 0} \binom{-n}{k}(-t)^k = \sum_{k \geq 0} \binom{k+n-1}{n-1} t^k$

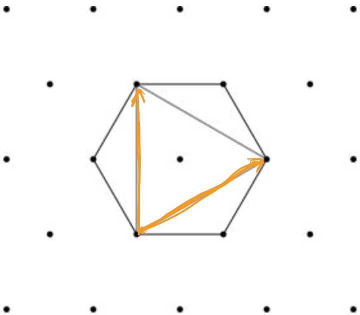
h^*-Polynomials

$\Lambda \subseteq \mathbb{R}^n$ lattice
$P \subseteq \mathbb{R}^n$ d-dimensional
$h^*_P(t)$ of degree $\leq d$

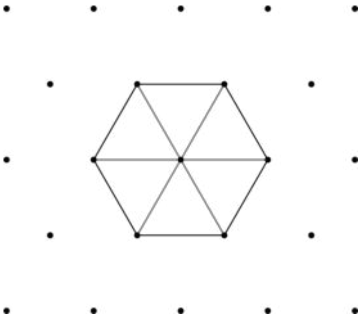
How to compute $h^*_P(t)$?

- using the definition

- using shellable unimodular triangulations



NOT unimodular



Unimodular

h^*-Polynomials

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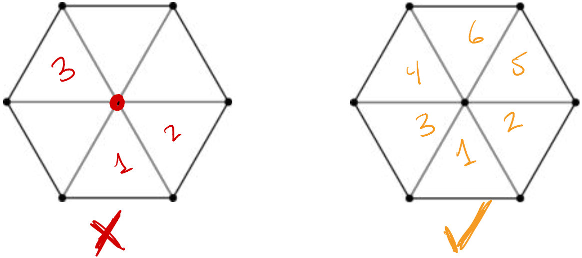

$c(T)$ = # facets in the intersection

[Stanley 1980]

$$h^*_P(t) = \sum_T t^{c(T)}$$

h^*-Polynomials

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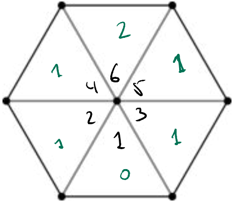


$C(T)$

$$h^*_P(t) = 1 + 4t + t^2$$

[Stanley 1980]

$$h^*_P(t) = \sum_T t^{c(T)}$$

h^*-Polynomials

$\Lambda \subseteq \mathbb{R}^n$ lattice

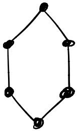
$P \subseteq \mathbb{R}^n$ d-dimensional

$h^*_P(t)$ of degree $\leq d$

$h^*_P(t) = \sum_T t^{c(T)}$

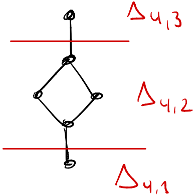
- If P is a unimodular simplex $\Rightarrow$ $h^*_P(t) = 1$
- Triangulate $[0,1]^n$ by slicing it w/ hyperplanes $x_i = x_j$

dual graph of triangulation:



$h^*_{[0,1]^3}(t) = 1 + 4t + t^2$

- First slice in hypersimplices and then refine using alcoves



h^*-Polynomials

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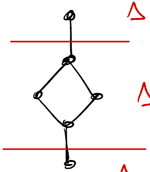


$\Delta_{4,3}$

$\Delta_{4,2}$

$\Delta_{4,1}$

ACR

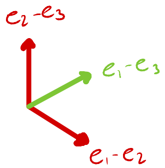
Theorems [Li 2012]

$$h^*_{\Delta'_{n,k}}(t) = \sum_{\substack{w \in \mathfrak{S}_{n-1} \\ exc(w) = k-1}} t^{des(w)} = \sum_{\substack{w \in \mathfrak{S}_{n-1} \\ des(w) = k-1}} t^{cover(w)}$$

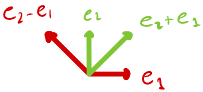
Φ - HYPERSIMPLICES

$$\underbrace{\{\alpha_1, \dots, \alpha_n\}}_{\text{simple}} \subseteq \Phi^+ \subseteq \Phi \quad \text{crystallographic root system}$$

$\{e_i - e_j : i \neq j\}$ $\qquad$ $\{e_i \pm e_j : i \neq j\} \cup \{\pm e_i\}$ $\qquad$ $\{e_i \pm e_j : i \neq j\} \cup \{\pm 2e_i\}$



type A_2



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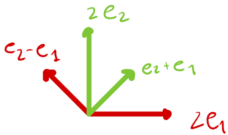


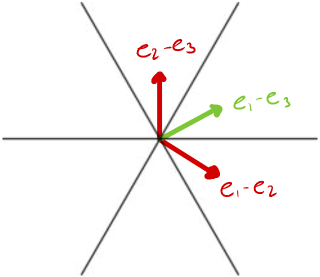
type C_2

Φ - Hypersimplices

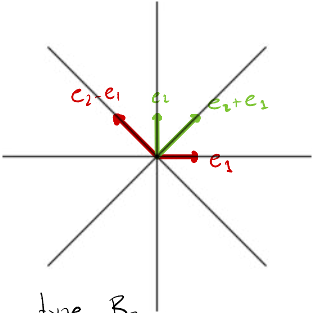
$\{\alpha_1, \ldots, \alpha_n\} \subseteq \Phi^+ \subseteq \Phi$ crystallographic root system

Linear arrangement $\mathcal{H}_\Phi = \{\alpha^\perp : \alpha \in \Phi^+\}$

Coxeter group W_Φ generated by reflections across $\alpha^\perp$

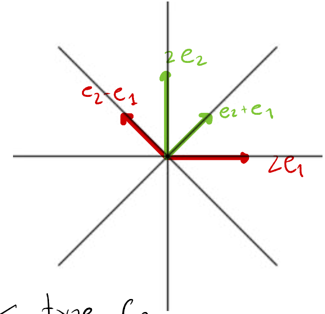


type A_2 S_n



type B_2

B_n



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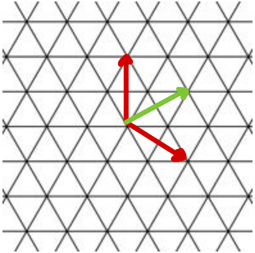
Φ - HYPERSIMPLICES

$\{\alpha_1, \dots, \alpha_n\} \subseteq \Phi^+ \subseteq \Phi$ crystallographic root system

Affine arrangement $\widetilde{\mathcal{H}}_\Phi = \{ H_{\alpha,k} : \alpha \in \Phi^+, k \in \mathbb{Z}\}$

$H_{\alpha,k} = \{ x \in V : \langle x, \alpha \rangle = k \}$

alcoves



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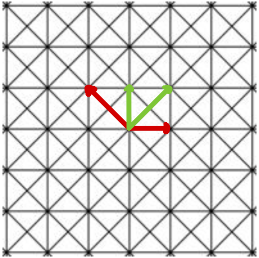
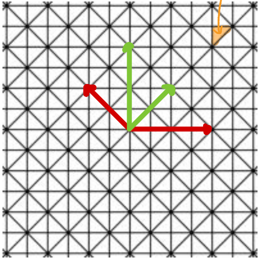


type B_2

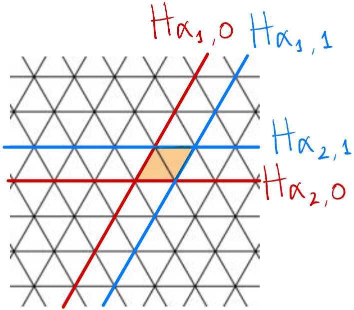


type C_2

$2x_1 = 1$

Φ - HYPERSIMPLICES

Fundamental parallelepiped $\Pi_\Phi := \{ x \in V \ : \ \forall\, i,\ 0 \leq \langle x, \alpha_i \rangle \leq 1 \}$



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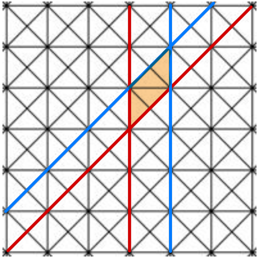
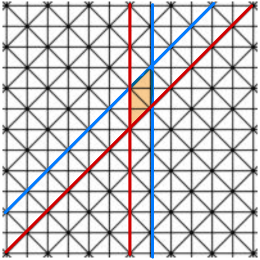


type B_2



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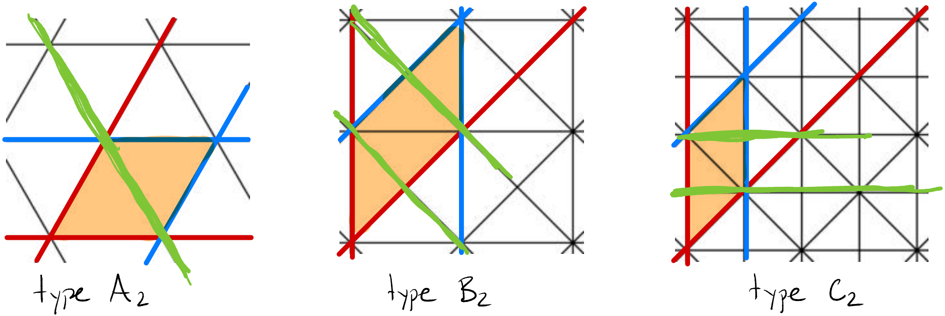
Φ - Hypersimplices

Fundamental parallelepiped $\Pi_\Phi := \{ x \in V \;:\; \forall i,\ 0 \le \langle x, \alpha_i \rangle \le 1 \}$

[Lam-Postnikov 2018]

Φ-hypersimplices : slice Π_Φ with hyperplanes $H_{\theta, k}$ highest root



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[Lam-Postnikov 2018]

Φ-hypersimplices : slice Π_Φ with hyperplanes $H_{\theta,k}$

Theorem (Lam-Postnikov)

$$\mathrm{Vol}(\Delta_{\Phi,k}) = \frac{1}{f} \# \{ w \in W_\Phi : \mathrm{cdes}(w) = k \}$$

index of conection of Φ

$$f = \frac{\# W_\Phi}{\mathrm{vol}\ \Pi_\Phi}$$

where $\mathrm{cdes}(w) = d_0(w) + a_1 d_1(w) + \cdots + a_n d_n(w)$ $\qquad \theta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

$$d_i(w) = \begin{cases} 1 & \text{if } w(\alpha_i) \in \Phi^- = -\Phi^+ \\ 0 & \text{else} \end{cases}$$ and $\alpha_0 = -\theta$

Φ - Hypersimplices

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[Lam-Postnikov 2018]

Φ-hypersimplices : slice Π_Φ with hyperplanes $H_{\theta, k}$

- Φ-hypersimplices are <u>alcoved polytopes</u> (triangulated)

  ↳ [Bullock-Jiang 2024] Ehrhart series

- In type A & C, alcoves are unimodular w.r.t. lattice generated by the vertices of $\Delta_{\Phi,1}$

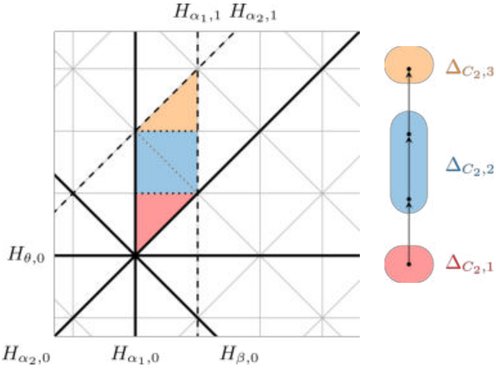
- Any linear extension of the weak order on the alcoves contained in Π_Φ defines a shelling. [Reading 2005]

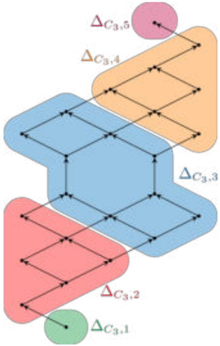
Type C Hypersimplices

$\boxed{\alpha_1 = 2e_1 \quad \alpha_2 = e_2 - e_1 \quad \cdots \quad \alpha_n = e_n - e_{n-1}}$ $\quad \theta = 2e_n$

$$\Delta_{C_n,k} = \{ x \in \mathbb{R}^n : 0 \leq 2x_1, x_2 - x_1, \ldots, x_n - x_{n-1} \leq 1, \ k-1 \leq 2x_n \leq k \}$$

$$\Delta_{C_n,1} = \mathrm{Conv}\left\{ 0, \tfrac{1}{2}e_n, \tfrac{1}{2}(e_{n-1}+e_n), \ldots, \tfrac{1}{2}(e_1+e_2+\cdots+e_n) \right\} \qquad \Lambda = \tfrac{1}{2}\mathbb{Z}^n$$





Combinatorial Model

$W_{C_n} = B_n$: signed permutations $\quad w = w_1 w_2 \cdots w_n$, $\quad w_i \in [\![n]\!] = \{\pm 1, \pm 2, \dots, \pm n\}$

such that $\quad |w_1|\,|w_2| \cdots |w_n| \in \mathfrak{S}_n$

complete notation $\quad \tilde{w} = \overline{w_n} \cdots \overline{w_2}\ \overline{w_1}\ w_1\, w_2 \cdots w_n$

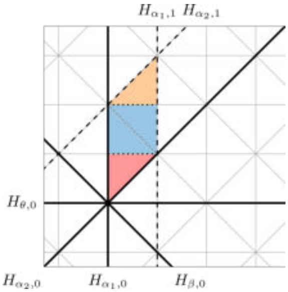
$$\mathrm{CDes} = \{ i \in [\![n]\!] : w_i > w_{i^+} \}$$

cyclically next in $\bar{n} \cdots \bar{2}\ \bar{1}\ 1\ 2 \cdots n$

Example: $\quad \bar{5}\ \bar{1}\ 4\ 2\ \bar{3}\ 3\ \bar{2}\ \bar{4}\ 1\ 5$

$$B_2 = \left\{ \begin{matrix} \bar{2}\,\bar{1}\,1\,2 & 2\,\bar{1}\,1\,\bar{2} & \bar{1}\,2\,\bar{2}\,1 & \bar{1}\,\bar{2}\,2\,1 \\ 1\,2\,\bar{2}\,\bar{1} & \bar{2}\,1\,\bar{1}\,2 & 1\,\bar{2}\,2\,\bar{1} & 2\,1\,\bar{1}\,\bar{2} \end{matrix} \right\}$$

$(f = 2)$



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such that $|w_1|\,|w_2| \cdots |w_n| \in \mathfrak{S}_n$

For $a,b \in [\![n]\!]$, write $a \ll b$ if $\exists\, c \in [\![n]\!]$ s.t. $a < c < b$

$\cancel{-1 \ll 1}$

Combinatorial Model

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For $a, b \in [\![n]\!]$, write $a \ll b$ if $\exists\, c \in [\![n]\!]$ s.t. $a < c < b$

$$\mathrm{BAsc}(w) = \{ i \in \{\bar{1}\} \cup [n] : w_i \ll w_{i+1} \}$$

Examples $\quad \bar{3}\ 3\ \bar{2}\ \bar{4}\ 1\ 5\ \bar{5} \qquad \bar{1}\ 1\ 2\ 3\ 4\ 5\ \bar{5} \qquad \bar{1}\ 1\ 2\ 3\ 4\ \bar{5}\ 5$

Combinatorial Model

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Let $X_n := \{ w \in B_n \ :\ w^{-1}(1) > 0 \}$ and

$u \longrightarrow w$ if u undoes a basc of w

$\bar{3}\ 3\ \bar{2}\ \bar{4}\ 1\ 5\ \bar{5}$

$\bar{3}\ \bar{2}\ \bar{4}\ 1\ 5 \qquad 3\ \bar{2}\ 1\ \bar{4}\ 5 \qquad 3\ \bar{2}\ \bar{4}\ 5\ 1$

Combinatorial Model

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$u \longrightarrow w$ if u undoes a basc of w

Thm (Abram, B.)

These are the cover relations of a poset (on X_n) isomorphic to the weak order on the alcoves contained in Π_{C_n}.

Moreover w corresponds to an alcove in $\Delta_{C_n, \mathrm{cdes}(w^{-1})}$

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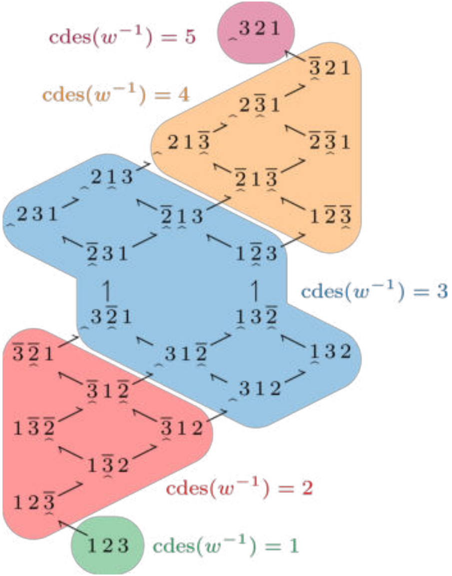
$X_3 =$



Combinatorial Model

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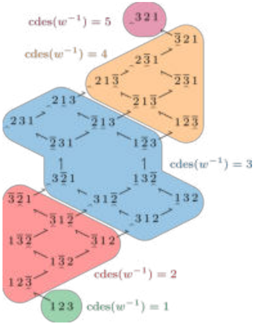

Explicit isomorphism $w \longmapsto A(w)$

Combinatorial Model

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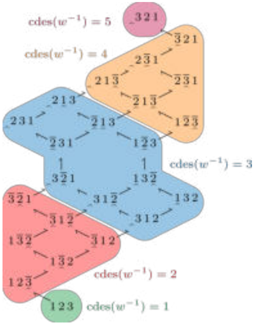

Explicit isomorphism $w \longmapsto A(w)$

proof by example

$w = 4\,\bar{2}\,1\,\bar{3}\,5$ $\qquad \mathrm{CDes}(w^{-1}) = \{1, 2, 5, \bar{2}, \bar{3}\}$

$v^6 \in \mathbb{Z}^5 \; : \; v_i^6 = \#\,[i-1] \cap \mathrm{CDes}(w^{-1})$

$v^6 = (0, 1, 2, 2, 2)$

$v^k = v^{k+1} + \frac{1}{2} e_{w_k}$

$v^5 = (0, 1, 2, 2, \frac{5}{2})$

$v^4 = (0, 1, \frac{3}{2}, 2, \frac{5}{2})$

$v^3 = (\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2})$

$v^2 = (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 2, \frac{5}{2})$

$v^1 = (\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2})$

$A(w) = \mathrm{Conv}\{v^1, v^2, \ldots, v^{n+1}\}$

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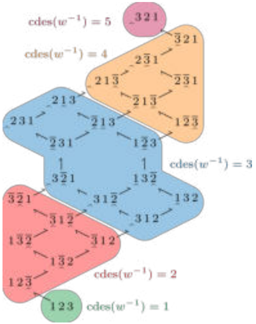
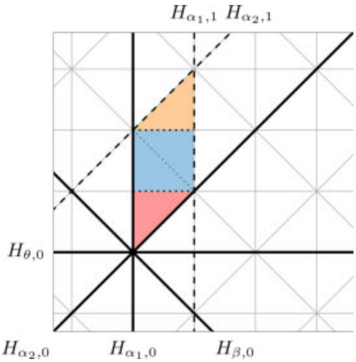


w	$\mathrm{cDes}(w^{-1})$	v^3	v^2	v^1
$2\ 1$	$\{1, 2, \bar{2}\}$	$(0, 1)$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, \frac{3}{2})$
$\bar{2}\ 1$	$\{1, \bar{2}\}$	$(0, 1)$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, \frac{1}{2})$
$1\ \bar{2}$	$\{1, \bar{2}\}$	$(0, 1)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
$1\ 2$	$\{2\}$	$(0, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$



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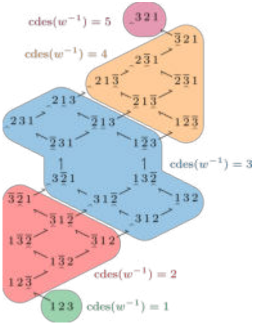

Consequence: first formula

The half-open hypersimplices are: $\Delta'_{C_n,1} = \Delta_{C_n,1}$ and, for $k \geq 2$,

$$\Delta'_{C_n,k} = \{ x \in \mathbb{R}^n : 0 \leq 2x_1, x_2 - x_1, \dots, x_n - x_{n-1} \leq 1, \ k-1 < 2x_n \leq k \}$$

Theorem (Abram, B.)

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w \in X_n \\ \mathrm{cdes}(w^{-1}) = k}} t^{\mathrm{basc}(w)}$$

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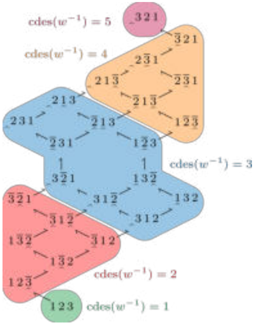

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$(X_n, \rightarrow)$ will return...

Generating Functions

$$\Delta'_{C_n,k} = \left\{ x \in \mathbb{R}^n : 0 \le \underbrace{2x_1}_{y_1}, \underbrace{x_2 - x_1}_{\frac{1}{2}y_2}, \dots, \underbrace{x_n - x_{n-1}}_{\frac{1}{2}y_n} \le 1 \ , \ k-1 < 2x_n \le k \right\}$$

$$r\Delta'_{C_n,k+1} = \left\{ y : 0 \le y_1 \le r, \ 0 \le y_2, \dots, y_n \le 2r, \ kr < y_1 + \dots + y_n \le (k+1)\, r \right\}$$

Generating Functions

$$\Delta'_{C_n,k} = \Big\{ x \in \mathbb{R}^n : 0 \le \underbrace{2x_1}_{y_1}, \underbrace{x_2 - x_1}_{\frac{1}{2}y_2}, \dots, \underbrace{x_n - x_{n-1}}_{\frac{1}{2}y_n} \le 1, \quad k-1 < 2x_n \le k \Big\}$$

$$\# r\Delta'_{C_n,k+1} \cap \tfrac{1}{2}\mathbb{Z}^n = \#\Big\{ y \in \mathbb{Z}^n : 0 \le y_1 \le r, \ 0 \le y_2, \dots, y_n \le 2r, \ kr < y_1 + \dots + y_n \le (k+1)r \Big\}$$

$$= \left([x^{kr+1}] + \dots + [x^{(k+1)r}] \right) (1 + x + \dots + x^r)(1 + x + \dots + x^{2r})^{n-1}$$

$$= \dots = [x^{kr}] \frac{(1-x^r)(1-x^{r+1})(1-x^{2r+1})^{n-1}}{x^r (1-x)^{n+1}}$$

$$\rightsquigarrow \quad \sum_{n\ge 1} \sum_{k \ge 0} L_{\Delta'_{C_n,k+1}}(r)\, s^k u^n = \frac{(1-us^2)^{r+1} - (1-u)^{r+1}}{(1+s)\left((1-u)^{r+1} - s(1-u)(1-us^2)^r\right)}$$

Generating Functions

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[Foata-Han 2009]

$$\sum_{n\geq 0}\sum_{w\in\mathfrak{B}_n} s^{\mathrm{fexc}(w)} t^{\mathrm{fdes}(w)} \frac{u^n}{(1-t^2)^n} = \sum_{r\geq 0}\left(\frac{(1-s)(1-t)t^{2r}}{(1-u)^{r+1}(1-us^2)^{-r}-s(1-u)} - \frac{(1-s)(1-t)t^{2r+1}}{(1-u)^{r+1}(1-us^2)^{-r}-s(1-us)}\right)$$

where $\mathrm{fexc}(w) = 2\,\#\{i\in[n-1] : w_i > i\} + \#\{i\in[n] : w_i < 0\}$

$\mathrm{fdes}(w) = 2\,\#\{i\in[n-1] : w_i > w_{i+1}\} + \delta_{w_1<0}$

$$\mathrm{des}_W(w) = \left\lfloor \frac{\mathrm{fdes}(w)+1}{2} \right\rfloor$$

[Abram, B.]

$$\sum_{n\geq 1}\sum_{w\in X_n} s^{\mathrm{fexc}(w)} t^{\mathrm{des}_W(w)} \frac{u^n}{(1-t)^{n+1}} = \frac{1}{(1+s)}\sum_{r\geq 0}\frac{(1-us^2)^{r+1}-(1-u)^{r+1}}{(1-u)^{r+1}-s(1-u)(1-us^2)^r}t^r$$

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Theorem (Abram, B.)

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w\in X_n \\ \mathrm{fexc}(w)=k-1}} t^{\mathrm{des}_W(w)}$$

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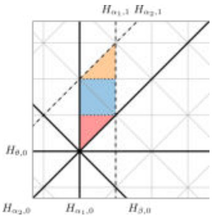
$$h^*_{\Delta'_{C_{n,k}}}(t) = \sum_{\substack{w\in X_n \\ \mathrm{fexc}(w)=k-1}} t^{\mathrm{des}_W(w)}$$

w	exc	neg	fexc	des_W
$2\ 1$	1	0	2	1
$\bar{2}\ 1$	0	1	1	1
$1\ \bar{2}$	0	1	1	1
$1\ 2$	0	0	0	0

Corollary 1:

$$\mathrm{Vol}(\Delta_{C_{n,k}}) = \#\{w\in X_n : \mathrm{fexc}(w)=k-1\}$$



$$h^*_{\Delta_{C_{2,1}}}(t) = 1$$

$$h^*_{\Delta_{C_{2,2}}}(t) = 2t$$

$$h^*_{\Delta_{C_{2,3}}}(t) = t$$

Corollary 2:

$$B_{n,k} = \mathrm{Vol}(\Delta_{C_{n,2k-1}}) + 2\,\mathrm{Vol}(\Delta_{C_{n,2k}}) + \mathrm{Vol}(\Delta_{C_{n,2k+1}})$$

More on the poset X_n

Let $\Psi_{C_n}(t) = \sum_{w \in X_n} t^{\operatorname{basc}(w)}$

More on the poset X_n

Let $\psi_{C_n}(t) = \sum_{w \in X_n} t^{\mathrm{basc}(w)}$

Recall: $A_n(t) := \sum_{w \in \mathfrak{S}_n} t^{\mathrm{des}(w)}$

$$A_{n+1}(t) = (1+nt)A_n(t) + t(1-t)A_n'(t)$$

$$\sum_{k \geq 0} (k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}$$

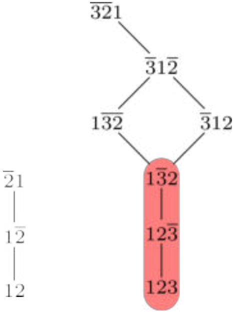
$$\mathrm{Eul}_A(t,x) := \sum_{n \geq 0} A_n(t) \frac{x^n}{n!}.$$

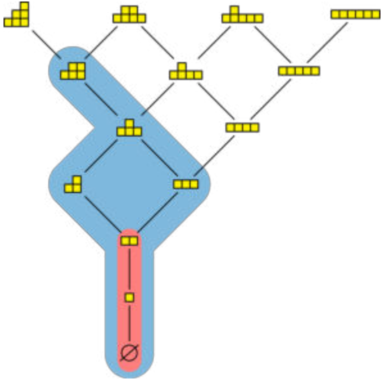
Proposition

- $\psi_{C_{n+1}}(t) = (1+(2n+1)t)\psi_{C_n}(t) + 2t(t-1)\psi_{C_n}'(t)$
- $\sum_{k \geq 0} (k+1)(2k+1)^{n-1} t^k = \dfrac{\psi_{C_n}(t)}{(1-t)^{n+1}}$
- $\sum_{n \geq 0} \psi_{C_{n+1}}(t) \dfrac{x^n}{n!} = e^{3x(t-1)} \mathrm{Eul}_A(t,2x)^2$
- $\psi_{C_n}(t)$ is real-rooted
- $\psi_{C_n}(t) + t^n \psi_{C_n}(t^{-1}) = B_n(t)$

More on the poset X_n

Theorem (Abram, B.) The "limit" of the posets X_n is the lattice of strict partitions.





[Abram, Chapelier-Laget, Reutenauer 2021] Analogous result in type A

Types B and D

- $\Lambda = \mathbb{Z}\{\text{vertices of } \Delta_{\Phi,1}\}$
  - Not all alcoves are lattice polytopes w.r.t. Λ
  - Λ is not $\widetilde{W}_{\Phi}$ invariant

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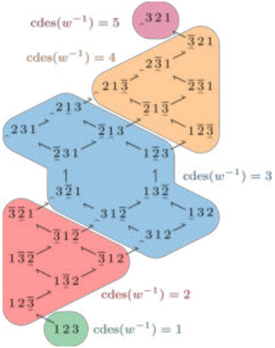
  - Not all alcoves are lattice polytopes w.r.t. Λ
  - Λ is not $\widetilde{W}_\Phi$ invariant

Can still define $\Psi_{B_n}(t)$ and $\Psi_{D_n}(t)$ (no longer h^*-polynomials)

n	$\Psi_{B_n}(t)$	$\Psi_{D_n}(t)$
3	$1+15t+7t^2+t^3$	$1+4t+t^2$
4	$1+56t+102t^2+32t^3+t^4$	$1+22t+18t^2+6t^3+t^4$
5	$1+189t+898t^2+706t^3+125t^4+t^5$	$1+85t+222t^2+138t^3+33t^4+t^5$
6	$1+610t+6351t^2+10876t^3+4751t^4+450t^5+t^6$	$1+294t+1895t^2+2380t^3+1047t^4+142t^5+t^6$

Conjecture: $\Psi_{B_n}(t) + t^n \Psi_{B_n}(t) = 2D_n(t)$

RECAP



- New statistic (basc) and partial order X_n:
- Two formulas for $h^*_{\Delta'_{C_n,k}}$:

$$h^*_{\Delta'_{C_n,k}}(t) = \sum_{\substack{w\in X_n \\ \mathrm{cdes}(w^{-1})=k}} t^{\mathrm{basc}(w)} = \sum_{\substack{w\in X_n \\ \mathrm{fexc}(w)=k-1}} t^{\mathrm{des}_w(w)}$$

- Recursion + generating function + real-rootedness of

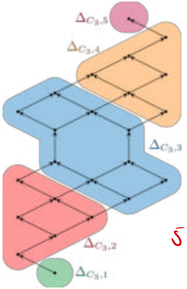
$$\Psi_{C_n}(t) := \sum_{w\in X_n} t^{\mathrm{basc}(w)}$$

THANK YOU!

Closed Hypersimplices

Proposition

$$h^*_{\Delta_{C_n,k}}(t) = h^*_{\Delta'_{C_n,k}}(t) + \sum_{j\geq 1}(1-t)^j\left(h^*_{\Delta'_{C_n,k-2j+1}}(t) + h^*_{\Delta'_{C_n,k-2j}}(t)\right)$$



$3t+3t^2$

$5t+t^2$

$$3t+3t^2+(1-t)(t+2t)+(1-t)^2$$
$$= 1+4t+t^2$$

$$5t+t^2+(1-t)1 = 1+4t+t^2$$