Higher-Categorical Associahedra

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Catalan Numbers

The Catalan numbers are a sequence $\{C_n\}$ of natural numbers where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

for $n \geq 0$.

The Catalan numbers enumerate many (at least 200) families of objects such as

- planar binary trees
- triangulations of a polygon
- maximal parenthesizations of a word
- Dyck paths
- Skeletal paths (B.-Charbonneau-Loehr-O'Connor-Mullins-Warrington)
- ...

Bracketings

Definition

A bracketing of $[n]$ is a collection of pairs of parentheses placed in this string of integers.

Example: Let $n = 4$. The maximal bracketings are

1. (1(23))4
2. 1((23)4)
3. 1(2(34))
4. (12)(34)
5. ((12)3)4

The pentagonal associahedron

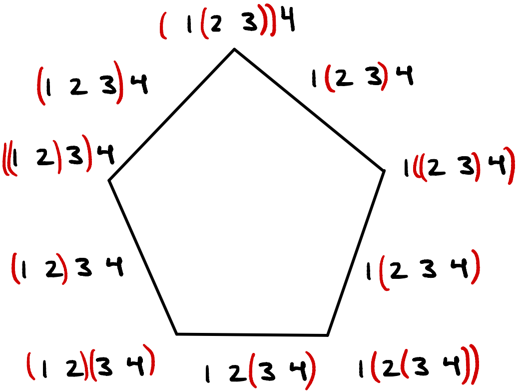


Figure: The bracketings of [4] arranged as a pentagon.

The Associahedron

Definition

The associahedron is a partially ordered set whose elements are the bracketings of $[n]$ ordered by containment.

The associahedron was independently discovered by Dov Tamari and Jim Stasheff. Stasheff's motivation came from studying the associativity of H-spaces.

The associahedron can be realized geometrically.

Associahedra as polytopes

Theorem (Many people)

The associahedron can be realized as the face poset of a polytope.

Realizations are due to

- Milnor (?)
- Haiman
- Lee
- Gelfand-Kapranov-Zelevinsky (secondary polytopes)
- Buchstaber, Shnider-Sternberg, Loday, Postnikov (generalized permutahedra)
- Chapoton-Fomin-Zelevinsky (cluster algebras)
- Rote-Santos-Steinu
- Black-De Loera-Lütjeharms-Sanyal (pivot rule polytopes)
- many others...

A realization of the associahedron

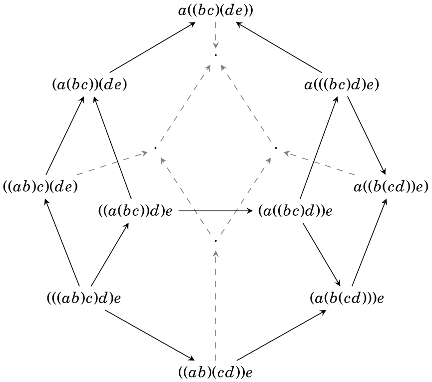


Figure: A 3-dimensional associahedron [Wikipedia]

Loday's realization of the associahedron

Loday's realization

- Inequalities: $\sum_{i=j}^{k} x_i \geq \binom{k-j+1}{2}$ for $1 \leq j \leq k \leq n-1$
- Equality: $\sum_{i=1}^{n-1} = \binom{n}{2}$.

For the pentagon we take

- $(1,0,0)^T x \geq 1$
- $(1,1,0)^T x \geq 3$
- $(0,1,0)^T x \geq 1$
- $(0,1,1)^T x \geq 3$
- $(0,0,1)^T x \geq 1$
- $(1,1,1)^T x = 6$

Loday's realization of the associahedron

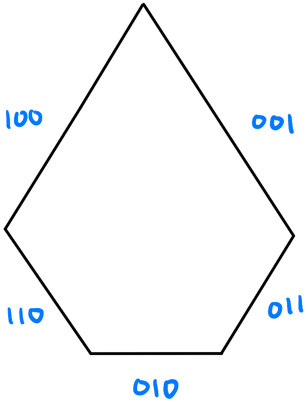


Figure: Loday's realization of the 2-dimensional associahedron

Loday's realization of the associahedron

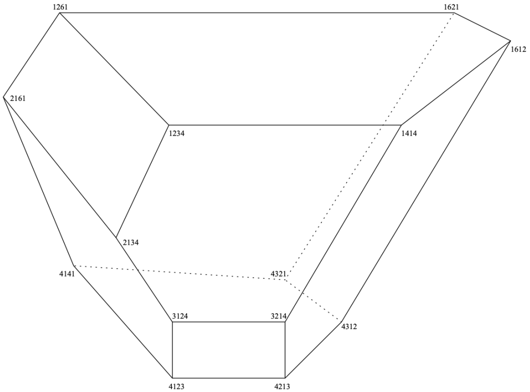


Figure: Loday's realization of the 3-dimensional associahedron [Loday]

Associahedra and symplectic geometry

Classical connection to symplectic geometry

- Associahedra form an important example of what is called an operad.
- One may take a category over an operad.
- An A_∞-category is a category over the associahedron.
- The Fukaya category of a symplectic manifold is the collection of Lagrangian submanifolds together with morphisms given by Floer cochains.
- The Fukaya category is an A_∞-category.

Associahedra and Fukaya categories

2-associahedra and functors between Fukaya categories

- Fukaya categories famously play a central role in Kontsevich's homological mirror symmetry conjecture.
- Mau-Wehrheim-Woodward showed how to construct a functor between Fukaya categories from a Lagrangian correspondence.
- Nathaniel Bottman introduced a notion of 2-associahedra as a family of posets which can be used for defining $(A_\infty, 2)-$categories of which these functors between Fukaya categories are an example.

Associahedra from compactified moduli spaces

A classical compactified moduli space [Kapranov, Drinfeld]

- The associahedron = poset of strata of a compactification of the moduli space of points on a real line.

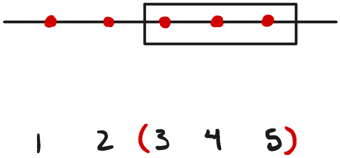


Figure: A collision of points on a line and the corresponding bracketing

2-associahedra from compactified moduli spaces

Compactified moduli spaces

- The associahedron = poset of strata of a compactification of the moduli space of points on a real line.
- Bottman proved that a 2-associahedron = poset of strata of a compactification of the moduli space of a collection of veritcal lines with marked points in $\mathbb{R}^2$.

We will use this model to give an intuitive combinatorial definition of 2-associahedra in terms of objects called 2-bracketings.

2-bracketing

Definition (codimension one 2-bracketing)

Let X be a collection of vertical lines in $\mathbb{R}^2$ together with points on these lines. A codimension one 2-bracketing B of X is either

- a collision of points on a line where the points form a consecutive set, or
- a collision of lines and points, i.e. a consecutive subset of lines and an ordered partition of the points on these lines which respects the order of the points on the individual lines.

A codimension one 2-bracketing

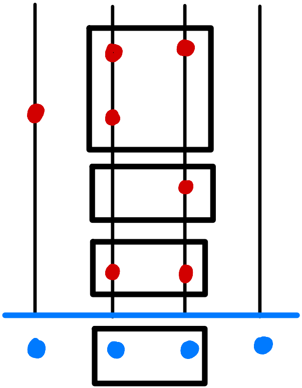


Figure: A codimension one 2-bracketing encoding a collision of lines and points

Definition (2-associahedra)

- A 2-bracketing is a union of compatible codimension one 2-bracketings.
- The 2-associahedron W_X determined by X is the collection of all 2-bracketings ordered by containment.

A 2-bracketing

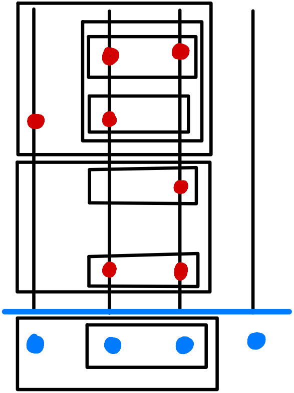


Figure: A 2-bracketing

2-associahedra as polytopes

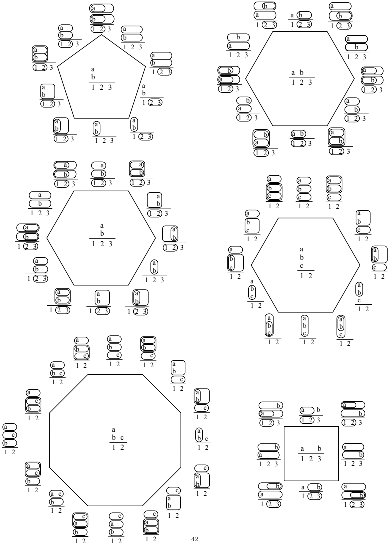
Conjecture (Bottman)

The 2-associahedra can be realized as face posets of convex polytopes.

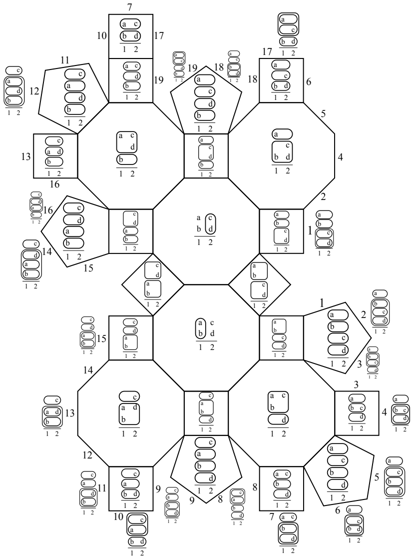
Partial results

- Bottman proved that they are abstract polytopes.
- Bottman-Mavrides proved they are Eulerian.
- Abouzaid-Bottman (using results of Bottman-Oblomkov) proved that they are homeomorphic to closed balls.

The 2-dimensional 2-associahedra



A 3-dimensional 2-associahedron



n-associahedra

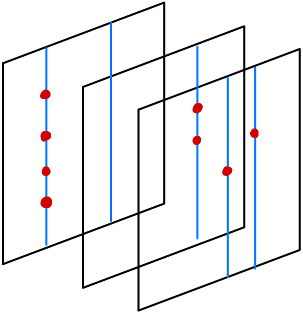
We introduce a natural extension of associahedra and 2-associahedra called categorical n-associahedra.

Definition (B.-Bottman-Poliakova)

Let X be an arrangement of affine coordinate flags in $\mathbb{R}^n$ whose incidence relations are encoded by a rooted plane tree $\mathcal{T}$.

- A codimension one n-bracketing is a combinatorial object which encodes a collision of subspaces forming a subtree in this arrangement.
- An n-bracketing is a union of compatible codimension one n-bracketings.
- The n-associahedron W_X is the collection of n-bracketings ordered by containment.

A tree arrangement of affine coordinate flags



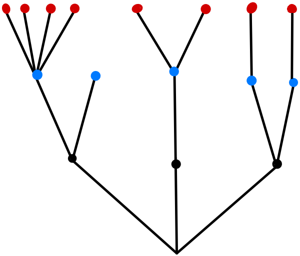


Figure: On the left is an arrangement of affine spaces for producing a categorical 3-associahedron. On the right is a rooted plane tree encoding the combinatorial type of the arrangement on the left.

A 2-dimensional 3-associahedron

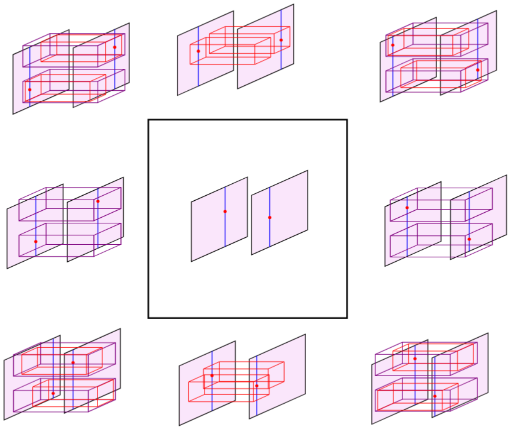


Figure: A 2-dimensional 3-associahedron

Our main result

Theorem (B.-Bottman-Poliakova)

There exists a family of complete fans whose face posets are the n-associahedra.

The velocity fan

For outlining the construction of our fan, we focus on the case of 2-associahedra.

Velocity fan

- Let $R(X)$ be a geometric arrangement of lines and points where the lines are one unit apart and the heights of the points increase by one unit as we move from left to right.
- Let B be a collision of points and lines and let $R(B)$ encode the arrangement of points and lines above after the collision.

A geometric arrangement of lines and points

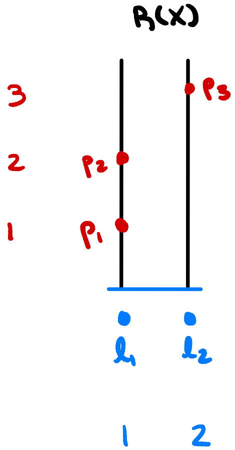


Figure: A geometric arrangement of lines and points

A geometric collision of lines and points

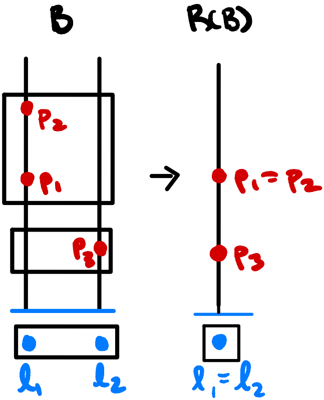


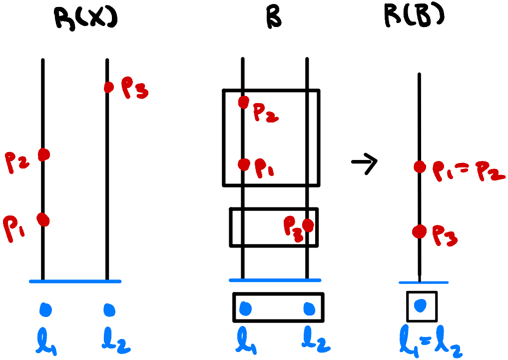
Figure: A collision and the resulting geometric arrangement of lines and points

Velocity fan

We construct a vector v_B associated to B:

- Let $v_B(x_i)$ encode the change in distance between lines l_i and l_{i+1} when passing from $R(X)$ to $R(B)$.
- Let $v_B(y_i)$ encodes the change in vertical distance from points p_i and p_{i+1} when passing from $R(X)$ to $R(B)$.

The calculation of a ray of the velocity fan



$$V_B = (1, 1, 2)$$

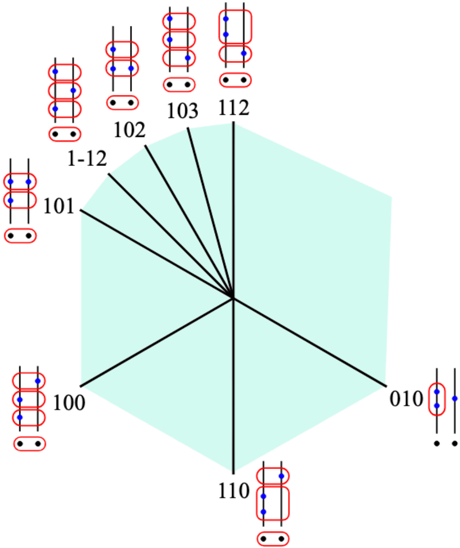
Figure: The calculation of a collision vector

Velocity fan

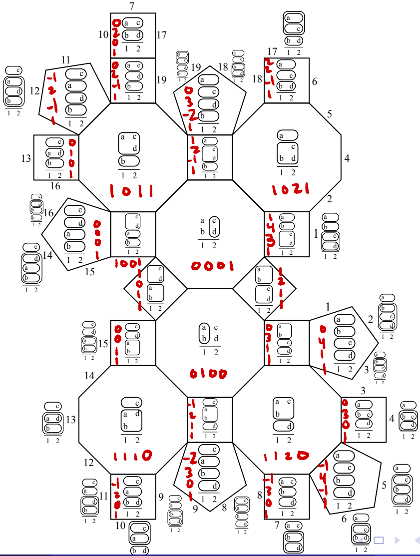
We construct a ray ρ_B associated to B:

- We let $\rho_B = \{\lambda v_B + \gamma\vec{1} : \lambda, \gamma \in \mathbb{R}, \lambda \geq 0\}$.
- For each 2-bracketing $\mathcal{B}$ of W_X, we associate the cone which is the convex hull of all ρ_B such that $B \in \mathcal{B}$.
- The velocity fan is the collection of all such cones.

The velocity fan of a 2-dimensional 2-associahedron



The velocity fan labeling of a 3-dimensional 2-associahedron



Loday's associahedron and Generalized permutahedra

When $n = 1$, our fan specializes to the normal fan of Loday's associahedron.

Some Remarks

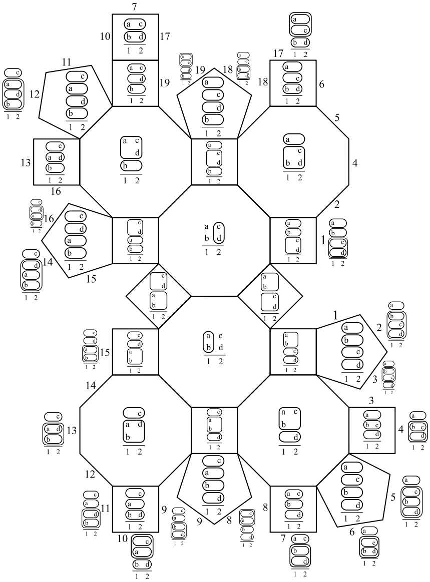
- Loday's associahedron is an example of a generalized permutahedron, equivalently a polymatroid. These are the polytopes whose normal fans coarsen the braid arrangement.
- It is impossible to realize all 2-associahedra as generalized permutahedra because their face posets are too large.
- Moreover, n-associahedra cannot be realized in any of the standard extensions of the braid arrangement.

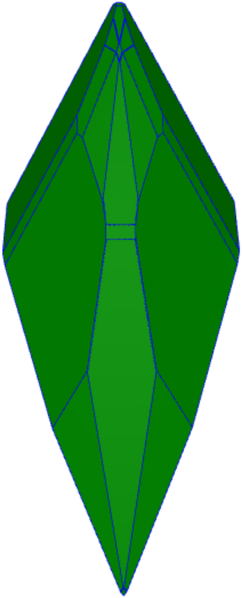
Proof of the complete fan property

Proof of fan property

- We introduce a notion of a metric n-bracketing.
- These are the conical combinations of compatible collisions and they can be equivalently described by a set of relations suggested by symplectic geometry.
- We prove that the collection of metric n-bracketings form a cone complex (a topological structure which arises naturally in logarithmic geometry), whose face poset is the corresponding n-associahedron.
- We produce a piecewise linear isomorphism from the cone complex of metric n-bracketings to the velocity fan.

We prove completeness by demonstrating combinatorially that the fan has no boundary.





Investigating the structure of the velocity fan

Results

- We produce a canonical unimodular flag triangulation of the velocity fan on the same set of rays.
- We demonstrate a piecewise-unimodular function on the velocity fan such that the image of each cone is a union of cones in the braid arrangement.
- We describe the extent to which the local nested fiber product structure of n-associahedra is realized by the velocity fan.
- We give a recursive calculation of the normal vectors of the walls of the velocity fan.
- For the class of *concentrated* n-associahedra we exhibit generalized permutahedra having velocity fans as their normal fans recovering Loday's associahedron and Forcey's multiplihedron.

Triangulating the velocity fan

Definition

Let $\{\mathscr{C}_i : 1 \leq i \leq k\}$ be a collection of collisions. Suppose that for all $1 \leq i < j \leq k$ either

1. $\mathscr{C}_i \sim \mathscr{C}_j$ or
2. $\mathscr{C}_i \to \mathscr{C}_j$,

then we say that the collection $\{\mathscr{C}_i\}$ is *nested*. $\triangle$

Theorem (B., Bottman, Poliakova)

The collection of nested collisions induces a canonical unimodular flag triangulation of the velocity fan on the same set of rays.

We also produce a second, finer unimodular flag triangulation, which provides a new generalization of the braid arrangement.

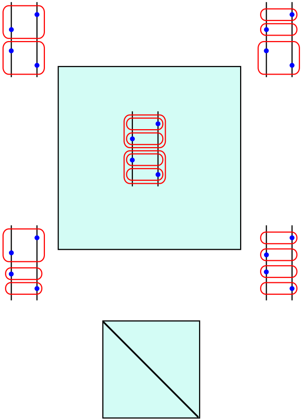


Figure: The triangulation of a square cone in the velocity fan.

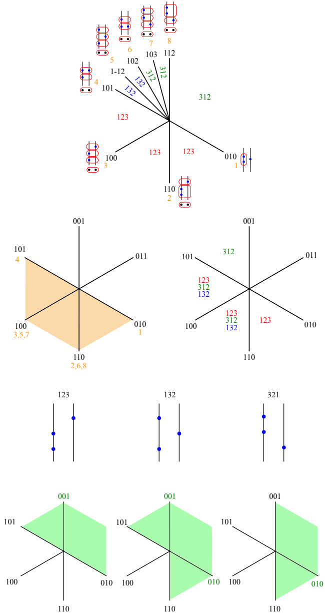
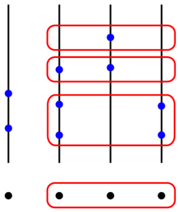


Figure: The velocity fan of $W_{2,1}$ and maps to the braid arrangement.

Local fiber product structure



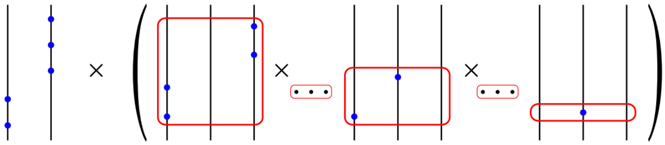


Figure: A collision $\mathscr{C}$ in a 2-associahedron, and the corresponding factorization of interval $\mathcal{K}(\mathcal{T})_{\mathscr{C}} = [\mathscr{C}, \star] \subseteq \mathcal{K}(\mathcal{T})$ into a smaller 2-associahedron $\mathcal{K}(\mathcal{T}/_{\mathscr{C}})$ on the left times the fiber product of three 2-associahedra over a single 1-associahedron $\mathcal{K}(\mathcal{T}|_{\mathscr{C}})$ on the right.

Thank You!