Kraśkiewicz-Hecke Insertion, a Type B Analogue to Hecke Insertion

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March 20, 2024

What is an Insertion Algorithm

Definition (Insertion Algorithm)

An algorithm that takes in data and outputs combinatorial objects such that there is a bijection between the set of all possible data and set of combinatorial objects.

Permutation

Definition (Permutation)

A *permutation* is some ordering of $[n] = \{1, 2, \ldots, n\}$. Let S_n denote the set of permutations.

Example

The elements of S_3 are:

- 123
- 132
- 213
- 231
- 312
- 321

Reduced Words

Definition (Simple Transposition)

A simple transposition, s_i, swaps positions i and $i+1$ of a permutation. S_n is generated by $s_1, \ldots, s_{n-1}$.

Definition (Reduced Word)

A reduced word of a permutation π is a minimal sequence of simple transpositions that compose into π.

Example

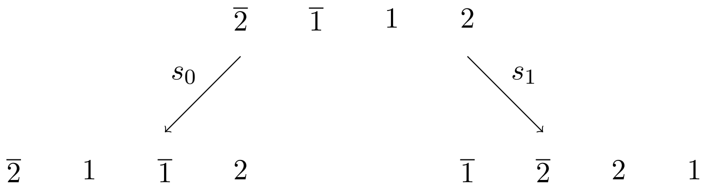
The reduced word(s) for each element of S_3 are:

- 123
- 132: s_2
- 213: s_1
- 231: $s_1 s_2$
- 312: $s_2 s_1$
- 321: $s_1 s_2 s_1$, $s_2 s_1 s_2$

Type B

Definition (Signed Permutations (Type B))

A *signed permutation* is a permutation, π, on $[-n] \cup [n]$ such that $\pi(i) = -\pi(-i)$. Signed permutations are generated by $s_0, s_1, \ldots, s_{n-1}$. Denote the set of signed permutations on $[-n] \cup [n]$ as B_n.



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Example

The elements of B_2 are:

- 12
- 21
- $\overline{1}2$
- $2\overline{1}$
- $1\overline{2}$
- $\overline{2}1$
- $\overline{12}$
- $\overline{21}$

Reduced Words

Definition (Reduced Word)

A *reduced word* of a signed permutation π is a minimal sequence of generators that produce π.

Example

The reduced words for $31\overline{2}$ are:

- 10212
- 10121
- 12012

Relations of Reduced Words

$$\begin{aligned} \text{Commutation} \quad & s_i s_j = s_j s_i \ \text{ if } |i-j| \geq 2, \\ \text{Short Braid} \quad & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ \text{ if } i \geq 1, \\ \text{Long Braid} \quad & s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0. \end{aligned}$$

Example

The reduced words for $31\overline{2}$ are:

- 10212
- 10121
- 12012

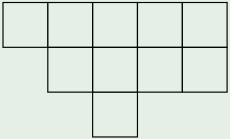
Partition

Definition (Strict Partition)

A *strict partition*, λ, of n is a sequence of k positive integers such that $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.

Example

$$\lambda = (5, 4, 1)$$



Standard Shifted Tableau

Definition

A *Standard Shifted Young Tableau* (SShT) is a shifted diagram filled with $\{1, \ldots, n\}$ such that rows are increasing rightwards and columns are increasing downwards.

1	2	3	6	7
	4	5	8	9
		10		

Definition

The *peak set* of a Standard Shifted Young Tableau is the set of entries i such that i is right of $i-1$ and above $i+1$.

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Unimodal

Definition (Unimodal)

A *unimodal* sequence is a sequence that is initially strictly decreasing, then strictly increasing.

$$a_1 > a_2 > \ldots > a_k < a_{k+1} < \ldots < a_\ell$$

Standard Decomposition Tableau

Definition

A *Standard Decomposition Tableau* (SDT) is a shifted tableau filled with nonnegative integers with rows $R_1, \ldots, R_\ell$ such that:

1. $R_\ell R_{\ell-1} \ldots R_2 R_1$ is a reduced word for a permutation π, call this sequence the reading word of the tableau.
2. R_i is a unimodal subsequence of maximum length in $R_\ell R_{\ell-1}, \ldots, R_{i+1} R_i$.

4	3	1	0	1	4
	3	2	0		
		1			

Kraśkiewicz Insertion

Kraśkiewicz insertion is a bijection from reduced words to pairs of Standard Decomposition and Standard Shifted Tableau of the same shape.

$$\{\text{Reduced words}\} \leftrightarrow \bigsqcup_{\lambda \vdash n \text{ strict}} \mathrm{SDT}(\lambda) \times \mathrm{SShT}(\lambda)$$

Example

$$0143 \leftrightarrow \begin{array}{|c|c|c|}\hline 4 & 1 & 3 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \; , \; \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

Properties of Kraśkiewicz Insertion

1. The reading word of the insertion tableau is the same permutation as the input.
2. The peak set of the input is the same as the peak set of the recording tableau.
3. The algorithm only depends on the insertion tableau.

Example

$0143 \leftrightarrow$

4	1	3
	0	

,

1	2	3
	4	

Kraśkiewicz Insertion Example

01432340

$\leftarrow 0$

Kraśkiewicz Insertion Example

01432340

0		1

Kraśkiewicz Insertion Example

01432340

| 0 | ← 1 |

| 1 |

Kraśkiewicz Insertion Example

01432340

0	1

1	2

Kraśkiewicz Insertion Example

01432340

Kraśkiewicz Insertion Example

01432340

0	1	4

1	2	3

Kraśkiewicz Insertion Example

01432340

0	1	4	← 3		1	2	3

Kraśkiewicz Insertion Example

01432340

Kraśkiewicz Insertion Example

01432340

4	1	3

$\leftarrow 0$

1	2	3

Kraśkiewicz Insertion Example

01432340

4	1	3
	0	

1	2	3
	4	

Kraśkiewicz Insertion Example

01432340

Kraśkiewicz Insertion Example

01432340

3 →

4	1	2
	0	

1	2	3
	4	

Kraśkiewicz Insertion Example

01432340

4	3	2
	0	← 1

1	2	3
	4	

Kraśkiewicz Insertion Example

01432340

4	3	2
	0	1

1	2	3
	4	5

Kraśkiewicz Insertion Example

01432340

4	3	2	3	4
	0	1		

1	2	3	6	7
	4	5		

Kraśkiewicz Insertion Example

01432340

4	3	2	3	4
	0	1		

$\leftarrow$ 0

1	2	3	6	7
	4	5		

Kraśkiewicz Insertion Example

01432340

3 →	4	3	2	0	4
		0	1		

1	2	3	6	7
	4	5		

Kraśkiewicz Insertion Example

01432340

4	3	2	0	4	
	0	1	← 2		

1	2	3	6	7
	4	5		

Kraśkiewicz Insertion Example

01432340

4	3	2	0	4
	0	1	2	

1	2	3	6	7
	4	5	8	

Kraśkiewicz Insertion Example

01432340

4	3	2	0	4
	0	1	2	

1	2	3	6	7
	4	5	8	

B_n Stable Schubert Polynomials

Definition ($\mathbf{a}$-compatible sequence)

Let $\mathbf{a}$ be a reduced word. A sequence of integers $(i_1, \ldots, i_n)$ is a $\mathbf{a}$-*compatible sequence* if

1. $i_1 \leq i_2 \leq \cdots \leq i_n$,
2. $i_j = \cdots = i_k$ only when $a_j \cdots a_k$ is a unimodal sequence.

Theorem (Billey-Haiman)

Let $G_w(\mathbf{x})$ be the B_n stable Schubert polynomial for the permutation w, let $R(w)$ be the set of reduced words for w, and let $K(\mathbf{a})$ be the set of $\mathbf{a}$-compatible sequences, then

$$G_w(\mathbf{x}) = \sum_{\mathbf{a}\in R(w)} \sum_{\mathbf{i}\in K(\mathbf{a})} 2^{\ell(\mathbf{i})-\ell_0(w)} x_{i_1} \cdots x_{i_n}$$

0-Hecke Expressions

Definition (0-Hecke Expression)

A 0-Hecke expression for a permutation π is a sequence of simple transpositions that compose into π, if you ignore the transpositions that do not increase the length of the permutation.

Example

Some 0-Hecke expressions for $\overline{2}31$ are:

- 102
- 120
- 1020
- 1022
- 1200
- 1202

0-Hecke Expression Relations

$$\begin{aligned} \text{(Annihilation)} \quad & s_i s_i = s_i \\ \text{Commutation} \quad & s_i s_j = s_j s_i \ \text{ if } |i-j| \geq 2, \\ \text{Short Braid} \quad & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ \text{ if } i \geq 1, \\ \text{Long Braid} \quad & s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0. \end{aligned}$$

Example

Some 0-Hecke expressions for $\overline{2}31$ are:

- 102
- 120
- 1020
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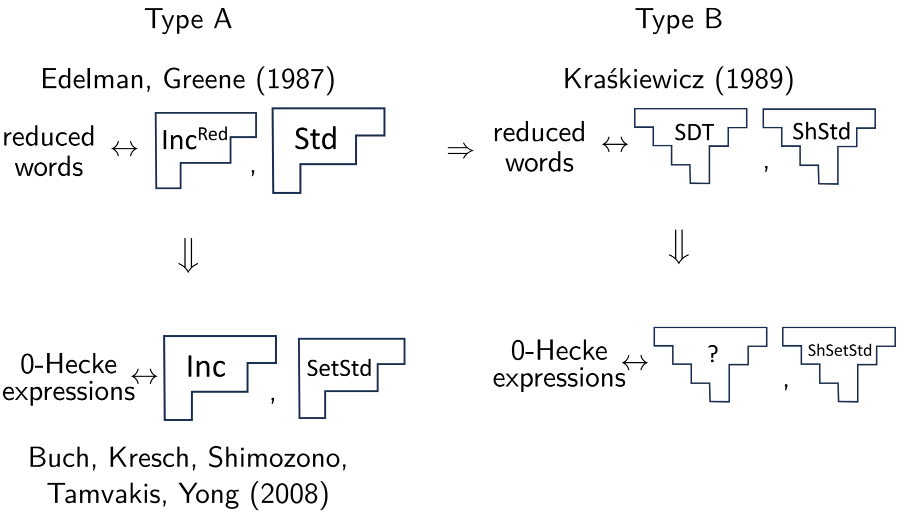
- 1200
- 1202

Diagram


Type A
Edelman, Greene (1987)
reduced words ↔ Inc^Red, Std
⇒
Type B
Kraśkiewicz (1989)
reduced words ↔ SDT, ShStd
⇓
0-Hecke expressions ↔ Inc, SetStd
Buch, Kresch, Shimozono, Tamvakis, Yong (2008)
⇓
0-Hecke expressions ↔ ?, ShSetStd

Properties of Kraśkiewicz Insertion

1. The reading word of the insertion tableau is the same permutation as the input.
2. The peak set of the input is the same as the peak set of the recording tableau.
3. The algorithm only depends on the insertion tableau.

Strict Decomposition Tableau

Definition

A shifted tableau P is a *Strict Decomposition Tableau* of π if:

1. $R_\ell R_{\ell-1} \dots R_1$ is a 0-Hecke expression for π.
2. Each row, R_i, is unimodal.
3. The first and last entries of R_{i+1} are less than the first entry of R_i for $i \in [k-1]$.
4. When considering sign if an element a is less than the element above it, b, then each value, v, in the reading word between a and b either satisfies $v \leq |a|$ and $v < |b|$ or satisfies $v \geq |a|$ and $v > |b|$.

Strict Decomposition Tableau Examples

4	1	0	3
	2	3	

4	1	0	3
	2	1	

4	0	1	3
	2	3	

4	3	1	3
	1	0	2

Strict Decomposition Tableau Examples

$\bar{4}$	$\bar{1}$	0	3
	$\bar{2}$	3	

$\bar{4}$	$\bar{1}$	0	3
	$\bar{2}$	$\bar{1}$	

$\bar{4}$	0	1	3
	$\bar{2}$	3	

$\bar{4}$	$\bar{3}$	$\bar{1}$	3
	$\bar{1}$	0	2

Strict Decomposition Tableau Examples

$\overline{4}$	$\overline{1}$	0	3
	$\overline{2}$	3	

$\overline{4}$	$\overline{1}$	0	3
	$\overline{2}$	$\overline{1}$	

$\overline{4}$	0	1	3
	$\overline{2}$	3	

$\overline{4}$	$\overline{3}$	$\overline{1}$	3
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Strict Decomposition Tableau Examples

$\overline{4}$	$\overline{1}$	0	3
	$\overline{2}$	3	

$\overline{4}$	$\overline{1}$	0	3
	$\overline{2}$	$\overline{1}$	

$\overline{4}$	0	1	3
	$\overline{2}$	3	

$\overline{4}$	$\overline{3}$	$\overline{1}$	3
	$\overline{1}$	0	2

Insertion Example

4	2	0	1	3	← 1
	2	0	1	3	

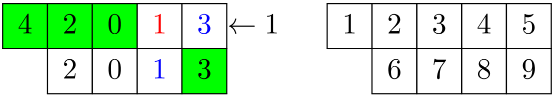
1	2	3	4	5
	6	7	8	9

Insertion Example

4	2	0	1	3	← 1
	2	0	1	3	

1	2	3	4	5
	6	7	8	9

Insertion Example



Insert the smallest number between 1 and 3 in the green region.
If no such number exists insert 3 instead.

Insertion Example

2 →	4	2	0	1	3
		2	0	1	3

1	2	3	4	5
	6	7	8	9

Insertion Example

$2 \rightarrow$	4	2	0	1	3
		2	0	1	3

1	2	3	4	5
	6	7	8	9

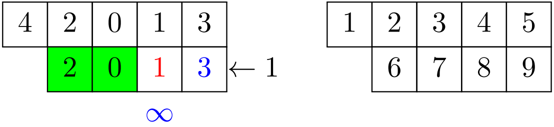
Insert the largest number between 0 and 2 in the green region.
If no such number exists insert 0 instead.

Insertion Example

4	2	0	1	3	
	2	0	1	3	$\leftarrow 1$

1	2	3	4	5
	6	7	8	9

Insertion Example



Insert the largest number between ∞ and 3 in the green region.
If no such number exists insert 3 instead.

Insertion Example

	4	2	0	1	3
3 →		2	0	1	3

1	2	3	4	5
	6	7	8	9

Insertion Example

4	2	0	1	3
	3	0	1	3
		2		

1	2	3	4	5
	6	7	8	9
		10		

Insertion Example

4	2	0	1	3	$\leftarrow 1$
	3	0	1	3	
		2			

1	2	3	4	5
	6	7	8	9
		10		

Insertion Example

4	2	0	1	3	$\leftarrow 1$
	3	0	1	3	
		2			

1	2	3	4	5
	6	7	8	9
		10		

Insertion Example

4	2	0	1	3
	3	0	1	3
		2	$\leftarrow$ 2	

1	2	3	4	5
	6	7	8	9
		10		

Insertion Example

4	2	0	1	3	
	3	0	1	3	
		2	$\infty \leftarrow 2$		

1	2	3	4	5
	6	7	8	9
		10		

Insert the largest number between ∞ and ∞ in the green region.
If no such number exists insert ∞ instead.

Insertion Example

4	2	0	1	3
	3	0	1	3
		2		

1	2	3	4	5
	6	7	8	9
		10,11		

Kraśkiewicz–Hecke Insertion

Theorem (A, Hawkes, Hamaker, Pan)

The insertion algorithm described, called Kraśkiewicz–Hecke insertion, gives a bijection between the set of 0*-Hecke expressions and pairs of strict decomposition tableau and shifted set valued tableau of the same shape.*

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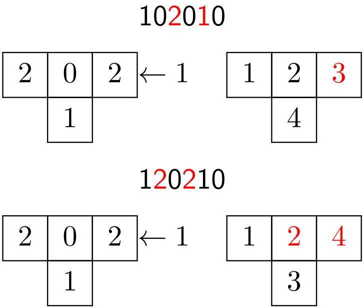
Inability to Preserve Peak Set



Definition

The *peak set* of a Standard Shifted Tableau is the set of entries i such that i is right of $i-1$ and above $i+1$.

Inability to Preserve Peak Set

Consider the set of words of length up to 5 equivalent to 10201:

$$10201 \rightarrow \begin{array}{|c|c|c|}\hline 2 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & 0 \\ \cline{2-3} \end{array}$$

$$10021 \rightarrow \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & 0 \\ \cline{2-3} \end{array}$$

$$12001 \rightarrow \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{2-3} \end{array}$$

$$12021 \rightarrow \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{2-3} \end{array}$$

$$12201 \rightarrow \begin{array}{|c|c|c|}\hline 2 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & 2 \\ \cline{2-3} \end{array}$$

$$10221 \rightarrow \begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & 0 \\ \cline{2-3} \end{array}$$

$$1201 \rightarrow \begin{array}{|c|c|c|}\hline 2 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

$$1021 \rightarrow \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}$$

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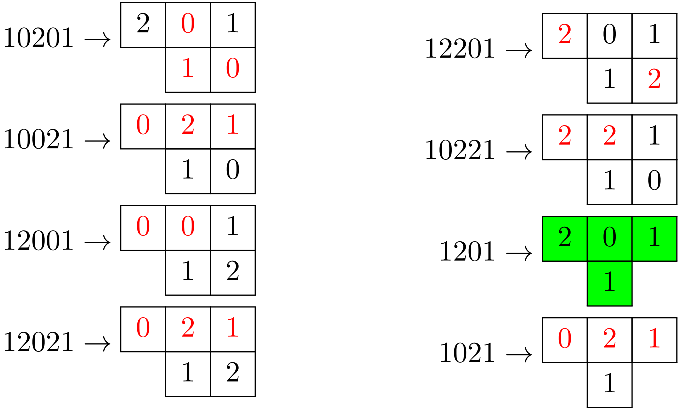
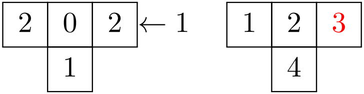
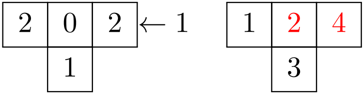


Inability to Preserve Peak Set

102010



120210



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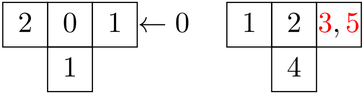
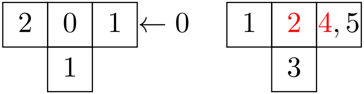
Inability to Preserve Peak Set

102010



120210



Definition

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Thanks

Thank you for listening!

Questions?

References

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