MODULAR RELATIONS BETWEEN CHROMATIC SYMMETRIC FUNCTIONS

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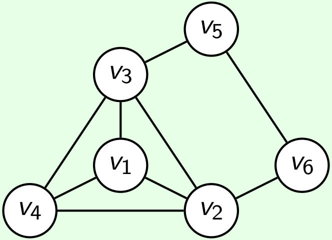
GRAPHS

A graph $G = (V, E)$ consists of a non-empty set of vertices V and a set of edges $E \subseteq \binom{V}{2}$.

- All graphs in this talk have finite sets of vertices.

Example



- $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$
- $E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_2v_6, v_3v_5, v_5v_6\}$

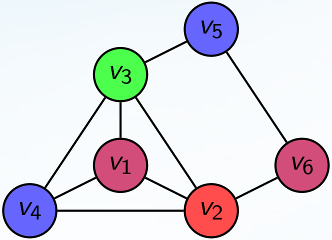
PROPER COLOURING

Given G with vertex set V a proper colouring κ of G in k colours is

$$\kappa : V \to \{1, 2, 3, \ldots, k\}$$

so if $v_i, v_j \in V$ are joined by an edge then

$$\kappa(v_i) \neq \kappa(v_j).$$



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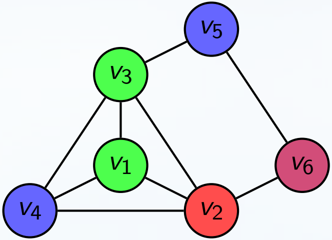


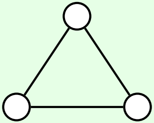
✗

CHROMATIC POLYNOMIAL: BIRKHOFF 1912

Given G the chromatic polynomial $\chi_G(k)$ is the number of proper colourings with k colours.

Example

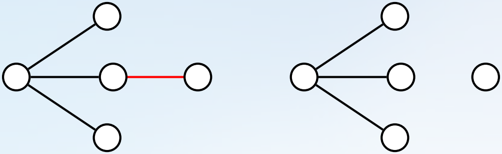
If G is the following graph



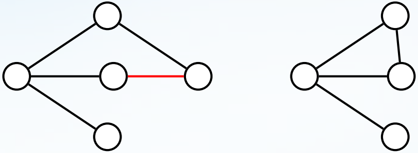
then $\chi_G(k) = k(k-1)(k-2)$.

DELETION-CONTRACTION

Delete ϵ: remove edge ϵ to get $G - \epsilon$.



Shrink ϵ: shrink edge ϵ and identify the vertices to get G/ϵ.



Theorem (Deletion-Contraction)

$$\chi_G(k) = \chi_{G-\epsilon}(k) - \chi_{G/\epsilon}(k)$$

Equivalently, $\chi_{G/\epsilon}(k) = \chi_{G-\epsilon}(k) - \chi_G(k)$.

PROPER COLOURING WITH INFINITELY MANY COLOURS

Given G with vertex set V a proper colouring κ of G is

$$\kappa : V \to \mathbb{Z}^+$$

so if $v_i, v_j \in V$ are joined by an edge, then

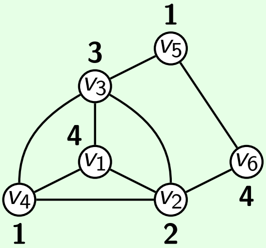
$$\kappa(v_i) \neq \kappa(v_j).$$

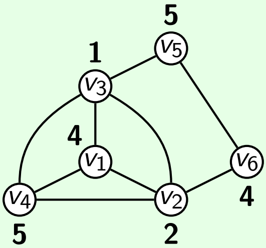
CHROMATIC SYMMETRIC FUNCTIONS: STANLEY 1995

Given a proper colouring κ of G on vertices $v_1, v_2, \ldots, v_n$ associate a monomial in commuting variables $x_1, x_2, x_3, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

Example





$$\prod_{i=1}^{6} x_{\kappa(v_i)} = x_4 x_2 x_3 x_1 x_1 x_4 = x_1^2 x_2 x_3 x_4^2$$

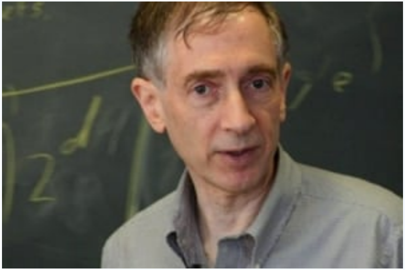
$$\prod_{i=1}^{6} x_{\kappa(v_i)} = x_4 x_2 x_1 x_5 x_5 x_4 = x_1 x_2 x_3 x_4^2 x_5^2$$

CHROMATIC SYMMETRIC FUNCTIONS: STANLEY 1995

Given G with vertices $v_1, v_2, \ldots, v_n$ the chromatic symmetric function of G is

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

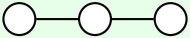
where the sum is over all proper colourings κ.



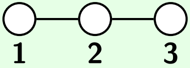
EXAMPLE

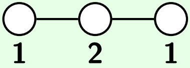
Example

Let G be the following graph.



Then the proper colourings of G are





and so on.

$$X_G = 6x_1x_2x_3 + x_1^2x_2 + \cdots$$

SYMMETRIC FUNCTIONS

Let $\mathfrak{S}_{(\infty)}$ be the group of all permutations σ of the set $\{1, 2, 3, \ldots\}$ which leave all but finitely many elements invariant; that is, $\sigma(i) \neq i$ for finitely many positive integers i.

A symmetric function is a formal power series f of bounded degree in commuting variables $x_1, x_2, \ldots$ such that for all permutations $\sigma \in \mathfrak{S}_{(\infty)}$,

$$f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$$

Let Sym be the set of all symmetric functions.

$$X_G \in \mathrm{Sym}$$

ELEMENTARY SYMMETRIC FUNCTIONS

An integer partition λ of n is a list $\lambda_1\lambda_2\dots\lambda_{\ell(\lambda)}$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$ and their sum is n.

$$544211 \vdash 17$$

The i-th elementary symmetric function is

$$e_i = \sum_{j_1<j_2<\cdots<j_i} x_{j_1}x_{j_2}\dots x_{j_i},$$

and for $\lambda = \lambda_1\lambda_2\dots\lambda_{\ell(\lambda)}$,

$$e_\lambda = e_{\lambda_1}e_{\lambda_2}\dots e_{\lambda_{\ell(\lambda)}}.$$

SYMMETRIC FUNCTIONS

Let

$$\mathrm{Sym}_n = \mathbb{Q}\text{-span}\{e_\lambda : \lambda \vdash n\}$$

Then

$$\mathrm{Sym} = \bigoplus_{n\geq 0} \mathrm{Sym}_n.$$

Proposition

Sym is a graded subalgebra of bounded degree power series, and $\{e_\lambda\}$ is a basis for it.

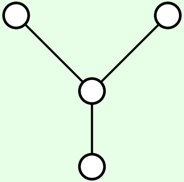
CHROMATIC SYMMETRIC FUNCTIONS IN TERMS OF e-BASIS

Example

If G is the path

$$X_G = 3e_3 + e_{21}$$

But if G is a claw,



$$X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4.$$

Which chromatic symmetric functions are e-positive?

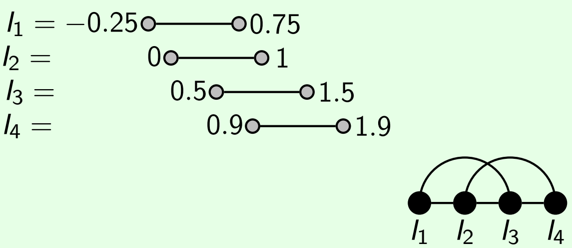
UNIT INTERVAL GRAPHS

Consider a set $\{I_1, I_2, \ldots, I_n\}$ and identify them with unit intervals

$$I_1 = [a_1, b_1], I_2 = [a_2, b_2], \ldots, I_n = [a_n, b_n]$$

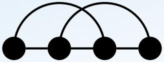
such that $a_1 \leq a_2 \leq \cdots \leq a_n$.
A unit interval graph is a graph with vertex set $\{\{I_1, I_2, \ldots, I_n\}\}$ such that I_i is adjacent to I_j if $I_i \cap I_j \neq \emptyset$.

Example



FIRST OPEN PROBLEM: STANLEY-STEMBRIDGE, GUAY-PAQUET

If G is the following unit interval graph



Then $X_G = 2e_{31} + 16e_4$.

Conjecture

Evey unit interval graph is e-positive.

SECOND OPEN PROBLEM

Stanley 1995:

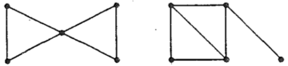


FIG. 1. Graphs G and H with $X_G = X_H$.

We do not know whether X_G distinguishes trees.

Heil and Ji 2018: The chromatic symmetric function distinguishes all trees up to 29 vertices.
(There are 5469566585 nonisomorphic trees on 29 vertices!)

Conjecture

If T and T' are trees. If $X_T = X_{T'}$, then $T \cong T'$.

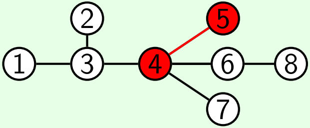
EQUIVALENT WITH RESPECT TO EDGES

We say (G, uv) is equivalent to $(H, u'v')$ written $(G, uv) \sim (H, u'v')$ if there is a bijection f from $V(G)$ to $V(H)$

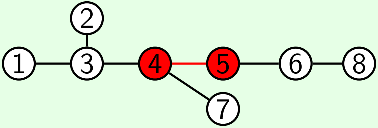
- $f(u) = u'$ and $f(v) = v'$, and
- $f : V(G/uv) \to V(H/u'v')$ is an isomorphism.

Example

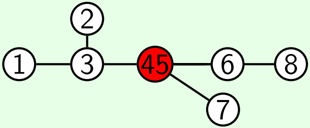
$(G, 45)$

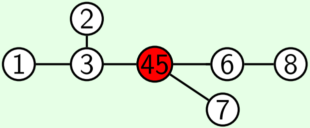


$(H, 45)$



Then $(G, 45) \sim (H, 45)$.

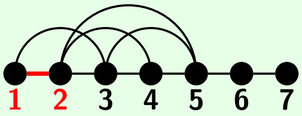




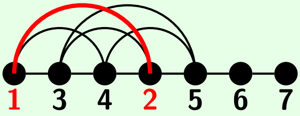
Another Example

Example

$(G, 12)$

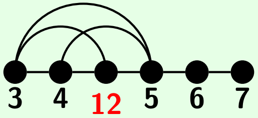


$(H, 12)$

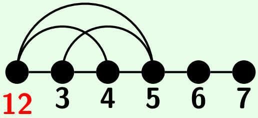


Then $(G, 12) \sim (H, 12)$.

$G/12$



$H/12$



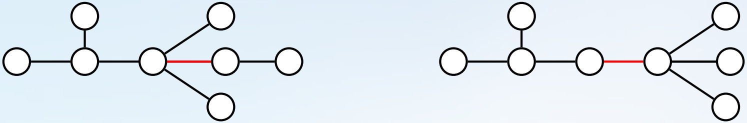
MODULAR RELATIONS

Theorem (A, Wang, and van Willigenburg 2020)

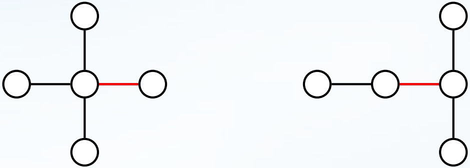
If $(G, uv) \sim (H, u'v')$, then

$$X_{G-uv} - X_G = X_{H-u'v'} - X_H.$$

EXAMPLE



We have $(G, \epsilon) \sim (H, \epsilon')$.



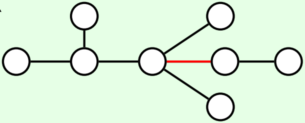
We have $(G, \epsilon) \sim (H, \epsilon')$.

EXAMPLE

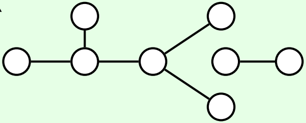
Example

$$X_{G-\epsilon} - X_G = X_{H-\epsilon'} - X_H \quad \text{or} \quad X_G = X_{G-\epsilon} - X_{H-\epsilon'} + X_H$$

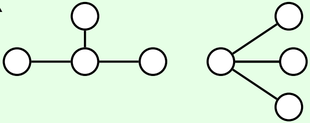
X



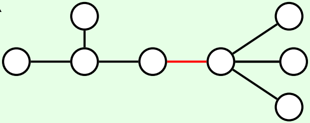
$=$ X



$-X$

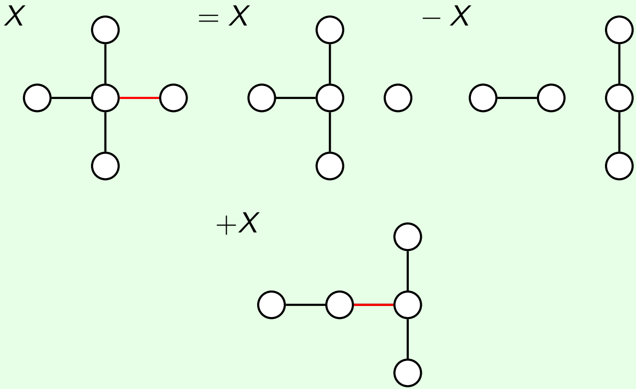


$+X$



EXAMPLE

Example



CHROMATIC BASES

$G_1 =$ 

Pick favourite simple connected graph on two vertices:

$G_2 =$ 

Pick favourite simple connected graph on three vertices:

$G_3 =$ 

And so on . . .

Let G_λ be the disjoint union $G_{\lambda_1} \cup \cdots \cup G_{\lambda_\ell}$.

Example

$G_{211} =$ 

CHROMATIC BASES

Theorem (Cho and van Willigenburg 2016)

$$\mathrm{Sym}_n = \mathbb{Q}\text{-span}\{X_{G_\lambda} : \lambda \vdash n\}$$

where

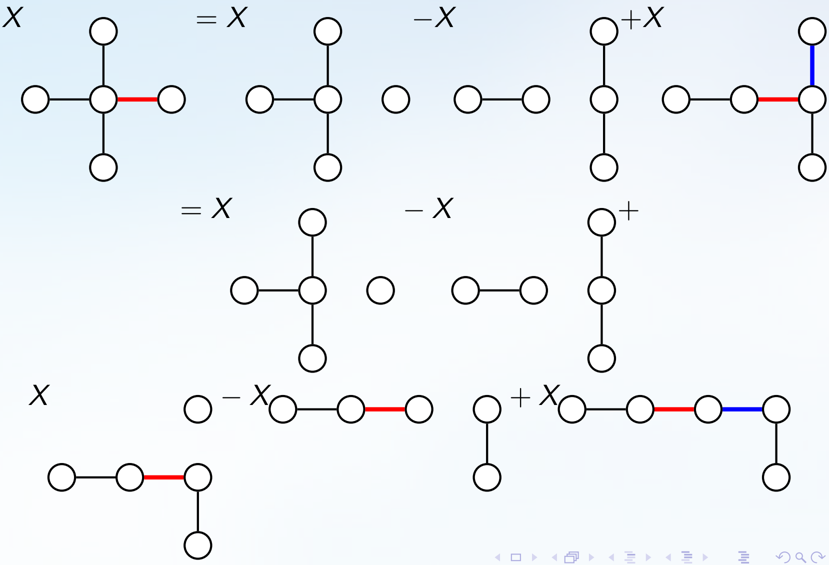
$$X_{G_\lambda} = X_{G_{\lambda_1}} \cdots X_{G_{\lambda_\ell}}.$$

Example

$G_{211} =$ 

$$X_{G_{211}} = X_{G_2} X_{G_1} X_{G_1}$$
$$= 2e_2 e_1 e_1 = 2e_{211}$$

TO PATHS



TO PATHS: ALGORITHM

INPUT: a tree T
OUTPUT: expansion of X_T in terms of path basis.

1. Set L a list with element (c, X_T)
2. Set $c = 1$.
2. **While** S contains (c, X_T), T not a path **do**
3. Choose one of its edges ϵ connected to a vertex of deg > 2.
4. In S, replace (c, X_T) by $(c, X_{T-\epsilon}), (-c, X_{T_\epsilon - \epsilon'}), (c, X_{T_\epsilon})$
5. Return $\sum_{(c, X_T) \in S} cX_T$.

e-POSITIVITY RESULTS

Theorem (A, Wang, and van Willigenburg 2021)

If $(G, \epsilon) \sim (H, \epsilon')$ and $G \cong H - \epsilon'$, then

$$X_G = \frac{X_H + X_{G-\epsilon}}{2}.$$

Consequently, if X_H and $X_{G-\epsilon}$ are e-positive so is X_G.

ARITHMETIC PROGRESSION

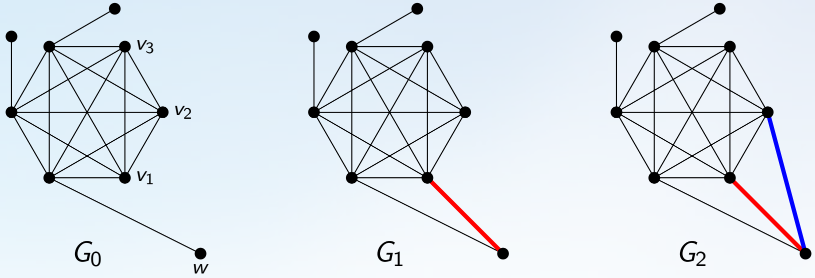
Theorem (A, Wang, and van Willigenburg 2021)

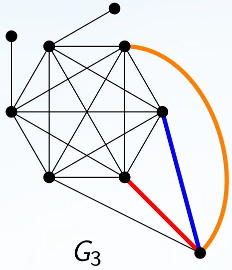
If $(G_1, \epsilon_1) \sim (G_2, \epsilon_2) \sim \cdots \sim (G_n, \epsilon_n)$ and $G_0 \cong G_1 - \epsilon_1$, $G_i \cong G_{i+1} - \epsilon_{i+1}$, then

$$X_{G_j} = \frac{k-j}{k} X_{G_0} + \frac{j}{k} X_{G_k} \quad \text{for all } 0 \leq j \leq k \leq n.$$

Consequently, if X_{G_0} and X_{G_k} are *e*- or *s*-positive so is X_{G_j}.

EXAMPLE





$$X_{G_1} = \frac{2}{3}X_{G_0} + \frac{1}{3}X_{G_3}.$$

EXAMPLE

$$X_{G_0} = 5760s_{(1^9)} + 7200s_{(2,1^7)} + 3168s_{(2^2,1^5)} + 468s_{(2^3,1^3)} + 2880s_{(3,1^6)}$$
$$+864s_{(3,2,1^4)} + 360s_{(4,1^5)},$$
$$X_{G_3} = 14400s_{(1^9)} + 12960s_{(2,1^7)} + 3888s_{(2^2,1^5)} + 288s_{(2^3,1^3)} + 2880s_{(3,1^6)}$$
$$+432s_{(3,2,1^4)},$$

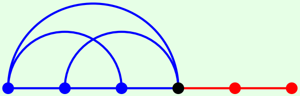
which are both Schur-positive.
The graph G_1 is also Schur-positive, since

$$X_{G_1} = \frac{2}{3}X_{G_0} + \frac{1}{3}X_{G_3} = 8640s_{(1^9)} + 9120s_{(2,1^7)} + 3408s_{(2^2,1^5)} + 408s_{(2^3,1^3)} +$$
$$2880s_{(3,1^6)} + 720s_{(3,2,1^4)} + 240s_{(4,1^5)}.$$

LOLLIPOP GRAPHS

When we identify a vertex of complete graph K_m with a vertex at the end of a path P_{n+1}, we have the lollipop graph $L_{m,n}$.

Example



$L_{4,2}$

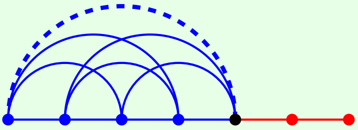
Theorem (Gebhard and Sagan 2001)

Lollipop graphs are e-positive.

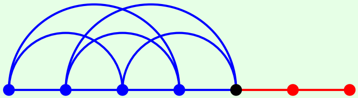
TYPE I MELTING LOLLIPOP GRAPH

The type I melting lollipop graphs $L_{m,n}^{(k)}$ for $m, n \geq 1$ and $0 \leq k \leq m-1$, obtained by deleting the edges between vertex m and vertices $1, \ldots, k$ from $L_{m,n}$.

Example



$L_{5,2}$



$L_{5,2}^{(1)}$

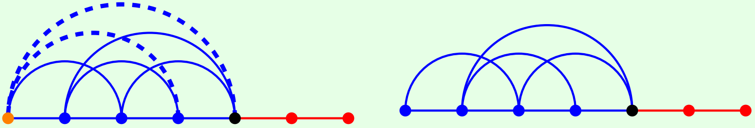
Theorem (Huh, Nam, and Yoo 2020)

Type I melting lollipop graphs are e-positive.

TYPE II MELTING LOLLIPOP GRAPH

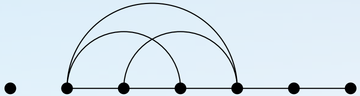
Type II melting lollipop graphs $\Gamma_{m,n}^{(k)}$ for $m \geq 3$, $n \geq 1$ and $1 \leq k \leq m-1$, obtained by deleting the edges between vertex 1 and vertices $m, \ldots, m-k+1$ from $L_{m,n}$.

Example

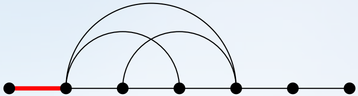


$L_{5,2}$ $\Gamma_{5,2}^{(2)}$

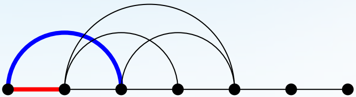
TYPE II MELTING LOLLIPOP GRAPHS ARE *e*-POSITIVE



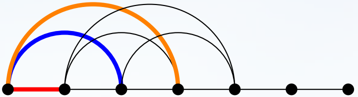
$K_1 L_{5-1,2}$



$\Gamma^{(3)}_{5,2}$



$\Gamma^{(2)}_{5,2}$



$L^{(1)}_{5,2}$

Theorem (A, Wang, and van Willigenburg 2021)

Type II melting lollipop graphs are *e*-positive.

SET PARTITIONS

A set partition π of $[n] = \{1, 2, \ldots, n\}$ is a set of disjoint sets $B_1, B_2, \ldots, B_\ell$ called blocks so that

- $B_i \neq \emptyset$
- $B_1 \cup B_2 \cup \cdots \cup B_\ell = [n]$.

$$\pi = B_1/B_2/\cdots/B_\ell \vdash [n]$$

Example

$$\{\{1,3,4\},\{2,5\},\{6\},\{7,8\}\}$$

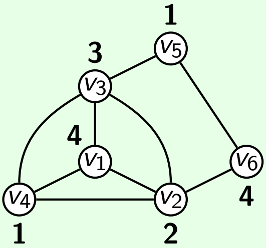
is a set partition of [8], or

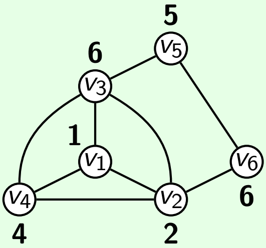
$$\pi = 134/25/6/78 \vdash [8].$$

CHROMATIC SYMMETRIC FUNCTIONS IN NON-COMMUTING VARIABLES: GEBHARD-SAGAN 2001

Given a proper colouring κ of vertices $v_1, v_2, \ldots, v_n$ and v_i labelled with i, associate a monomial in non-commuting variables $x_1, x_2, x_3, \ldots$, $x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$.

Example





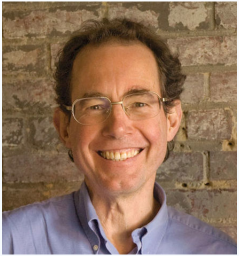
$\prod_{i=1}^{6} x_{\kappa(v_i)} = x_4 x_2 x_3 x_1 x_1 x_4$ $\qquad$ $\prod_{i=1}^{6} x_{\kappa(v_i)} = x_1 x_2 x_6 x_4 x_5 x_6$

CHROMATIC SYMMETRIC FUNCTIONS IN NON-COMMUTING VARIABLES: GEBHARD-SAGAN 2001

Given labelled G with vertices $v_1, v_2, \ldots, v_n$ the chromatic symmetric function of G in non-commuting variables $x_1, x_2, \ldots$ is

$$Y_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

where the sum over all proper colourings κ.

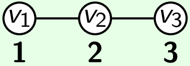


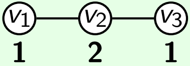
EXAMPLE

Example

Let G be the following graph.

Then the proper colourings of G are





and so on.

$$Y_G = x_1x_2x_3 + x_1x_2x_1 + \dots$$

NCSym

A symmetric function in non-commuting variable $x_1, x_2, \ldots$ is a formal power series f of bounded degree in non-commuting variables $x_1, x_2, \ldots$ such that for all permutations $\sigma \in \mathfrak{S}_{(\infty)}$,

$$f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots).$$

Let NCSym be the set of all symmetric functions in non-commuting variables.

$$Y_G \in \mathrm{NCSym}$$

ELEMENTARY FUNCTIONS IN NCSym

The elementary symmetric function in NCSym for $\pi \vdash [n]$ is

$$e_\pi = \sum_{(i_1, i_2, \ldots, i_n)} x_{i_1} x_{i_2} \cdots x_{i_n}$$

summed over all tuples $(i_1, i_2, \ldots, i_n)$ with

$$i_j \neq i_k$$

if j and k are in the same block of π.

Example

$e_{13/2} = x_1 x_1 x_2 + x_1 x_2 x_2 + x_2 x_2 x_1 + x_2 x_1 x_1 + \cdots + x_1 x_2 x_3 + \cdots$
and $\rho(e_{13/2}) = 2!1!e_{21}$

SYMMETRIC FUNCTIONS IN NON-COMMUTING VARIABLES

Let

$$\mathrm{NCSym}_n = \mathbb{Q}\text{-span}\{e_\pi : \pi \vdash [n]\}.$$

Then

$$\mathrm{NCSym} = \bigoplus_{n \geq 0} \mathrm{NCSym}_n.$$

Proposition

NCSym is a graded subalgebra of bounded degree power series in non-commuting variables, and $\{e_\pi\}$ is a basis for it.

PATHS ARE NOT e-POSITIVE

Let G be the following path graph.

Then

$$Y_G = \frac{1}{2}e_{123} + \frac{1}{2}e_{1/23} - \frac{1}{2}e_{13/2} + \frac{1}{2}e_{12/3}.$$

Even the paths are not e-positive in NCSym!

UBCSym

Given a set partition $\pi \vdash [n]$, define $\mathrm{type}(\pi)$ to be the pair (λ, a) where a is the size of the block containing n and the parts of λ are the sizes of the remaining blocks, e.g. $\mathrm{type}(1/24/35) = (21, 2)$. Let

$$T = \mathbb{Q}\text{-span}\{e_\pi - e_{\pi'} : \mathrm{type}(\pi) = \mathrm{type}(\pi')\}.$$

T is an ideal of NCSym.

Let

$$\overline{e_\pi} = e_\pi + T \in \mathrm{NCSym}/T := \mathrm{UBCSym}$$

and

$$\overline{Y_G} = Y_G + T \in \mathrm{UBCSym}.$$

$\overline{e}$-POSITIVITY, DAHLBERG 2019

Now we want to see which graphs are $\overline{e}$-positive, that is when $\overline{Y_G}$ can be written as a positive linear combination of $\{\overline{e_\pi}\}$.

Conjecture

Evey unit interval graph is $\overline{e}$-positive.

ARITHMETIC PROGRESSION

Theorem (A, Wang, and van Willigenburg 2021)

- If $(G,\epsilon) \sim (H,\epsilon')$ and $G \cong H - \epsilon'$ with some certain labelling, then

$$\overline{Y_G} = \frac{\overline{Y_H} + \overline{Y_{G-\epsilon}}}{2}.$$

Consequently, if $\overline{Y_H}$ and $\overline{Y_{G-\epsilon}}$ are $\overline{e}$-positive so is $\overline{Y_G}$.

- If $(G_1,\epsilon_1) \sim (G_2,\epsilon_2) \sim \cdots \sim (G_n,\epsilon_n)$ and $G_0 \cong G_1 - \epsilon_1, G_i \cong G_{i+1} - \epsilon_{i+1}$, with some certain labelling, then

$$\overline{Y_{G_j}} = \frac{k-j}{k}\overline{Y_{G_0}} + \frac{j}{k}\overline{Y_{G_k}} \quad \text{for all } 0 \leq j \leq k \leq n.$$

Consequently, if $\overline{Y_{G_0}}$ and $\overline{Y_{G_k}}$ are $\overline{e}$-positive so is $\overline{Y_{G_j}}$.

THANK YOU

Thank you ...