Involutive Groups from Graphs

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(Joint work with Alexander Wilson and Jonathan Bloom)

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Motivating Ideas

Definition (Striker '18)

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- Independent Sets of a graph

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- $\tau_1 = (\emptyset, \{1\})(\{3\}, \{1,3\})(\{4\}, \{1,4\})$
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Let $E = [4]$ and
$L = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}, \{4\}, \{3,4\}, \{2,3,4\}\}$

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- $\tau_4 = (\emptyset, \{4\})(\{1,2,3\}, \{1,2,3,4\})$

In this case $T_L \cong S_4 \rtimes C_2^3$

Natural Associated Poset Structure

To all generalized toggle groups, there is a natural partial order called the *toggle poset* which we denote by P_L.

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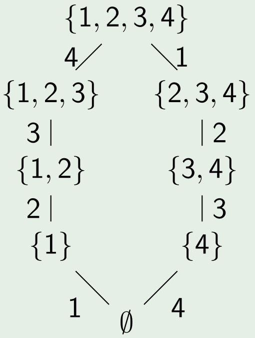
Example



Larger Family

All we had here was finite graphs (the Hasse diagrams) and a proper edge coloring (no two edges that share an endpoint have the same color).

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Definition

For $G = ([n], E)$ with proper surjective edge coloring $\kappa : E \to [i]$, we define the permutation group $\mathfrak{G}_\kappa$ as follows. For each $j \in [i]$, let $\tau_j = \prod_{\{a,b\} \in \kappa^{-1}(i)} (a, b)$ as an element of $\mathfrak{S}_n$. Then $\mathfrak{G}_\kappa$ is the group generated by the set of τ_j's.

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Example

$P_4, \kappa =$ 

$$\mathfrak{G}_\kappa = \langle (1,2)(3,4), (2,3) \rangle \cong D_4$$

What Groups can we get?

Everything if we don't impose any restrictions.

Theorem

Let $\mathfrak{G}$ be a finite group generated by involutions. Then there is a simple connected graph G with proper edge coloring κ such that $\mathfrak{G}_\kappa$ is isomorphic to $\mathfrak{G}$.

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Sketch of proof.

Consider the coloring κ of the Cayley Graph of $\mathfrak{G}$ from the presentation generated by involutions where each edge is colored by the generator used. This is proper because the generators are involutions. Additionally for each generator g_i, τ_{g_i} and g_i act identically, so the groups generated are isomorphic. □

Unfortunately, while the argument is constructive the resulting graphs are not helpful, as for instance $\mathfrak{S}_n$ the resulting graph has $n!$ vertices.

Centralizer

Definition

For a graph G of order n with proper edge coloring $\kappa : E(G) \to [k]$, the group $\mathrm{Aut}_\kappa(G) \subseteq S_n$ consists of all permutations $\sigma \in S_n$ such that $\{i, j\}$ is an edge of G colored a under κ if and only if $\{\sigma(i), \sigma(j)\}$ is an edge of G colored a under κ.

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For a graph G of order n with proper edge coloring κ, the centralizer $C_{S_n}(\mathfrak{G}_\kappa)$ of the corresponding group in the symmetric group is isomorphic to $\mathrm{Aut}_\kappa(G)$.

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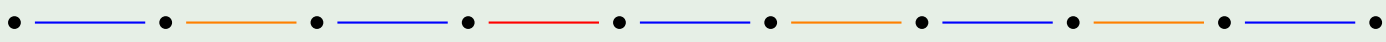
$P_{10}, \kappa =$ 

$P_8, \kappa' =$ 

This means that the center of $\mathfrak{G}_\kappa$ is contained within $\mathrm{Aut}_\kappa(G)$.

Classification of when the center is non-trivial for trees

Lemma

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Sketch.

Consider the vertices of maximum eccentricity. If unique, then $\mathrm{Aut}_\kappa(G)$ is trivial. If not unique and the rooted trees are not color isomorphic then $\mathrm{Aut}_\kappa(G)$ is trivial. If not unique and the trees are color isomorphic can swap the two of them, which forces the trees to just be swapped. □

Theorem

For G a tree and $\kappa : E(G) \to [k]$, either $Z(\mathfrak{G}_\kappa)$ is trivial, or $|G| = 2m$ is even and $\mathfrak{G}_\kappa \subseteq B_m$.

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This is an if and only if as every $\mathfrak{G}_\kappa$ on a tree contains a cycle on all vertices and in B_m any such cycle σ has that $\sigma^m \in Z(B_m)$.

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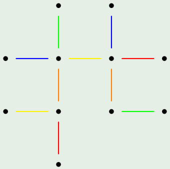

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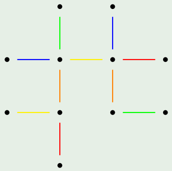

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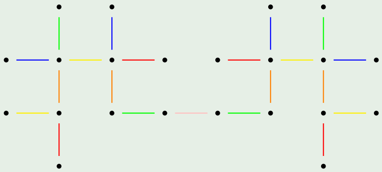

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Realizing the Symmetric Group

Trivially, we can always realize the Symmetric Group on any graph, if we just give every edge a distinct color. But how few do we need?

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If T is a tree and $\Delta(T)$ denotes the maximum degree of a vertex, we always need at most $\Delta(T) + 1$ colors.

Sketch.

Suppose κ is an edge coloring of T using $\Delta(T)$ colors and suppose ℓ is a leaf. Let κ' denote the coloring obtained by changing the edge adjacent to ℓ to a new color. Then $\mathfrak{G}_{\kappa'} \cong \mathfrak{S}_{|T|}$. □

In the case where $\Delta(T) \geq \frac{|T|}{2}$ then any coloring of T produces $\mathfrak{S}_{|T|}$.

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Let abc be a sequence of distinct, adjacent letters in a word w (with a or c allowed to be empty if b is the first or last letter respectively). We call abc a symmetric triple if there exists a subset S containing a and c of letters in w such that the graph obtained by deleting the edges labeled by elements of S in the path graph colored by w has exactly one component that has even order.

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Theorem

If $w = w_1 \dots w_\ell$ contains a symmetric triple abc, then $\mathfrak{S}_w \cong \mathfrak{S}_{\ell+1}$.

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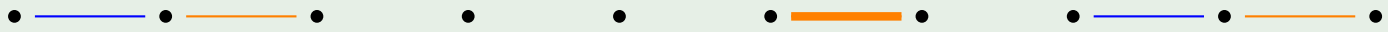
$P_{10}, \kappa =$ 

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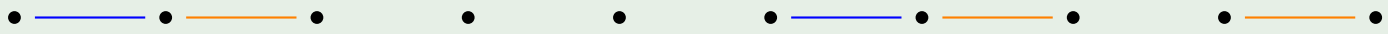
$P_{10}, \kappa =$ 

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Lemma

For a graph G with proper edge coloring κ, the group $\mathfrak{G}_\kappa$ is imprimitive if and only if there exists an imprimitive vertex coloring of G with respect to the edge coloring κ.

Examples

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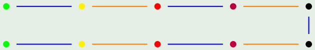

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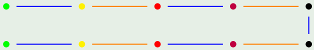

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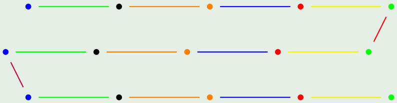

Toggles on Paths

Definition

The word w of a coloring κ of P_n is said to be a toggle word if for any consecutive subword w' of w with $|w'| > 1$ there are at least two letters that appear an odd number of times.

Importantly, this means that $\mathfrak{G}_\kappa$ is not a subgroup of A_n.

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- $\mathfrak{G}_\kappa \cong \mathfrak{S}_n$
- $PGL_d(q) \leq \mathfrak{G}_\kappa \leq P\Gamma L_d(q)$ *where* $n = \frac{q^d-1}{q-1}$ *for some prime power* q *and* $d \geq 2$.

Other Results/Conjectures

Theorem

If κ is a coloring of P_n, then for any $i, j \in [n]$ there is an involution in $\mathfrak{G}_\kappa$ that sends i to j.

Conjecture

If κ corresponds to a toggle word on P_n then $\mathfrak{G}_\kappa \cong \mathfrak{S}_n$

Theorem

If G is the Hasse Diagram of an Independence Poset of size n with κ the coloring of the edges induced by the vertices of the underlying Directed Acyclic Graph, then $\mathfrak{G}_\kappa$ contains A_n.

Conjecture

If G is the Hasse Diagram of a semidistrim lattice L of size n with κ the canonical labeling by the join irreducibles then $\mathfrak{G}_\kappa$ contains A_n.