Multiple Life Models

Lecture: Weeks 9-10

broken heart syndrome

\((T_x, T_y)\)
Chapter summary

- Approaches to studying multiple life models:
  - define multiple states
  - traditional approach (use joint random variables)

- Statuses:
  - joint life status
  - last-survivor status

- Insurances and annuities involving multiple lives
  - evaluation using special mortality laws

- Simple reversionary annuities

- Contingent probability functions

- Dependent lifetime models

- Chapter 9 (Dickson et al.)
Approaches multiple states

States in a joint life and last survivor model

\[
\begin{align*}
\mu_{x+t:y+t}^0 & : x \text{ alive} \\
\mu_{x+t:y+t}^1 & : x \text{ alive, } y \text{ dead} \\
\mu_{x+t:y+t}^2 & : x \text{ dead, } y \text{ alive} \\
\mu_{y+t}^3 & : x \text{ dead, } y \text{ dead}
\end{align*}
\]
Joint distribution of future lifetimes

Consider the case of two lives currently ages $x$ and $y$ with respective future lifetimes $T_x$ and $T_y$.

- Joint cumulative dist. function: $F_{T_x T_y}(s, t) = \Pr[T_x \leq s, T_y \leq t]$
  - independence: $F_{T_x T_y}(s, t) = \Pr[T_x \leq s] \times \Pr[T_y \leq t] = F_x(s) \times F_y(t)$

- Joint density function: $f_{T_x T_y}(s, t) = \frac{\partial^2 F_{T_x T_y}(s,t)}{\partial s \partial t}$
  - independence: $f_{T_x T_y}(s, t) = f_x(s) \times f_y(t)$

- Joint survival dist. function: $S_{T_x T_y}(s, t) = \Pr[T_x > s, T_y > t]$
  - independence: $S_{T_x T_y}(s, t) = \Pr[T_x > s] \times \Pr[T_y > t] = S_x(s) \times S_y(t)$
Illustrative example 1

Consider the joint density expressed by

\[ f_{T_x T_y}(s, t) = \frac{1}{64}(s + t), \quad \text{for } 0 < s < 4, \ 0 < t < 4. \]

1. Prove that \(T_x\) and \(T_y\) are not independent.

2. Calculate the covariance of \(T_x\) and \(T_y\).

3. Evaluate the probability \((x)\) outlives \((y)\) by at least one year.

Solution to be discussed in lecture.
\( f_x(s) = \int_0^4 \frac{1}{64} (s+t) \, dt = \frac{1}{16} (s+2), \quad 0 < s < 4 \)

\( f_y(t) = \int_0^4 \frac{1}{64} (s+t) \, ds = \frac{1}{16} (t+2), \quad 0 < t < 4 \)

\[
\frac{1}{16} (s+2) \cdot \frac{1}{16} (t+2) \neq \frac{1}{64} (s+t) \quad \text{not indep!}
\]

(b) \( \text{Cov}(T_x, T_y) = E(T_x T_y) - E(T_x) E(T_y) = \frac{-1}{9} \)

\[ E(T_x T_y) = \int_0^4 \int_0^4 s \cdot t \cdot \frac{1}{64} (s+t) \, ds \, dt = \frac{16}{3} \]

\[ E(T_x) = E(T_y) = \int_0^4 s \cdot \frac{1}{16} (s+2) \, ds = \frac{7}{3} \]
(a) \( P_r(T_x \geq T_y + 1) = \int_{0}^{3} \int_{t+1}^{4} \frac{1}{64} (s+t) \, ds \, dt \)

\[ \left[ \frac{1}{64} \left( \frac{s^2}{2} + st \right) \right]_{t+1}^{4} = \frac{28125}{5} = 5625 \]

This is probability \((x) \) within \((y)\) by at least 1
\[(x) \sim T_y\]

\[(T_x, T_y) \sim \{f, F, S\}\]

status

joint life

\[\text{last survivor} \quad \frac{\min(T_x, T_y) + \max(T_x, T_y)}{T_x + T_y}\]

April 27-29 out of town

no class

April 30 5-8 pm

Thursday

Room C 304
The joint life status

This is a status that survives so long as all members are alive, and therefore fails upon the first death.

- Notation: \((xy)\) for two lives \((x)\) and \((y)\)

- For two lives: \(T_{xy} = \min(T_x, T_y)\)

- Cumulative distribution function:

\[
F_{T_{xy}}(t) = \Pr[\min(T_x, T_y) \leq t] = 1 - \Pr[\min(T_x, T_y) > t] = 1 - \Pr[T_x > t, T_y > t] = 1 - S_{T_xT_y}(t, t) = 1 - tP_{xy}
\]

where \(tP_{xy} = \Pr[T_x > t, T_y > t] = S_{T_{xy}}(t)\) is the probability that both lives \((x)\) and \((y)\) survive after \(t\) years.
The case of independence

- Alternative expression for the distribution function:

\[ F_{T_{xy}}(t) = F_x(t) + F_y(t) - F_{T_x,T_y}(t,t) \]

- In the case where \( T_x \) and \( T_y \) are independent:

\[ t p_{xy} = \Pr[T_x > t, T_y > t] \]
\[ = \Pr[T_x > t] \times \Pr[T_y > t] \]
\[ = t p_x \times t p_y \]

and

\[ t q_{xy} = t q_x + t q_y - t q_x \times t q_y \]

- Remember this (even in the case of independence):

\[ t q_{xy} \neq t q_x \times t q_y \]
\[ t_{q \times y} = t_{q \times} \times t_{q y} \]

\[ \text{independent} \]

\[ t_{p \times y} = t_{p \times} \cdot t_{p y} \]

\[ t_{q \times y} = t_{q \times} \cdot t_{q y} \]

\[ t_{q \times y} = \left( t_{q x} \right) + \left( t_{q y} \right) - \left( t_{q x} \cdot t_{q y} \right) \]

\[ \frac{1}{a} \rightarrow t \]
The last-survivor status

This is a status that survives so long as there is at least one member alive, and therefore fails upon the last death.

- Notation: \((xy)\)
- For two lives: \(T_{xy} = \max(T_x, T_y)\)
- General relationship among \(T_{xy}, T_{xy}, T_x, \text{ and } T_y:\)

\[
T_{xy} + T_{xy} = T_x + T_y \\
T_{xy} \cdot T_{xy} = T_x \cdot T_y \\
aT_{xy} + aT_{xy} = aT_x + aT_y
\]

for any constant \(a > 0\).

- For each outcome, note that \(T_{xy}\) is equal either \(T_x\) or \(T_y\), and therefore, \(T_{xy}\) equals the other.
\[
\begin{align*}
\Pr_{x,y} = & \Pr(\max(T_x, T_y) > t) \\
= & \Pr(T_x > t) \Pr(T_y > t) = S(t, t)_{T_x, T_y}
\end{align*}
\]

Prob that \((x,y)\) survives to time \(t\)

\[
1 - t \Pr_{x,y} = t \Phi_{x,y}
\]
Distribution of $T_{xy}$

Recall method of inclusion-exclusion of probability:
\[ \Pr[A \cup B] + \Pr[A \cap B] = \Pr[A] + \Pr[B]. \]

- Choose events $A = \{T_x \leq t\}$ and $B = \{T_y \leq t\}$ so that
  \[ A \cup B = \{T_{xy} \leq t\} \quad \text{and} \quad A \cap B = \{T_{xy} \leq t\}. \]

- This leads us to the following useful relationships:
  \[
  \begin{align*}
  F_{T_{xy}}(t) + F_{T_{xy}}(t) &= F_x(t) + F_y(t) \\
  S_{T_{xy}}(t) + S_{T_{xy}}(t) &= S_x(t) + S_y(t) \\
  tP_{xy} + tP_{xy} &= tP_x + tP_y \\
  f_{T_{xy}}(t) + f_{T_{xy}}(t) &= f_x(t) + f_y(t)
  \end{align*}
  \]

- These relationships lead us to finding distributions of $T_{xy}$, e.g.
  \[ F_{T_{xy}}(t) = F_x(t) + F_y(t) - F_{T_{xy}}(t) = F_{T_{xy}}(t, t) \]
  which is obvious from $F_{T_{xy}}(t) = \Pr[T_x \leq t \cap T_y \leq t]$. 
\[ F_{T_{x,y}}(t) = \Pr(\max(T_x, T_y) \leq t) = \Pr(T_x \leq t, T_y \leq t) \]

\[ F_{T_{x,y}}(t, t) \]

\[ q_{x,y} = q_x + q_y - q_{x,y} \]

independence

\[ = q_x q_y + q_x q_y + q_x q_y \]

\[ 0 \quad t \]
Interpretation of probabilities

- Note that:
  - \( tP_{xy} \) is the probability that both lives \((x)\) and \((y)\) will be alive after \(t\) years.
  - \( tp_{xy} \) is the probability that at least one of lives \((x)\) and \((y)\) will be alive after \(t\) years.

- In contrast:
  - \( tq_{xy} \) is the probability that at least one of lives \((x)\) and \((y)\) will be dead within \(t\) years.
  - \( tq_{xy} \) is the probability that both lives \((x)\) and \((y)\) will be dead within \(t\) years.
Illustrative example 2

For independent lives \((x)\) and \((y)\), you are given:

\[
q_x = 0.05 \quad \text{and} \quad q_y = 0.10,
\]

and

\[
q_{x+1} = 0.06 \quad \text{and} \quad q_{y+1} = 0.12.
\]

Deaths are assumed to be uniformly distributed over each year of age. Calculate and interpret the following probabilities:

1. \(0.75q_{xy}\)  
   \[1 - 0.75p_{xy} = 1 - 0.75p_x \cdot 0.75p_y\]
   \[= 1 - (1 - 0.75(0.05))(1 - 0.75(0.1)) = 0.1096875\]

2. \(1.5q_{xy}\)  
   \[1.5q_x \cdot 1.5q_y = (1 - 0.5p_x \cdot 0.5p_{x+1})(1 - 0.5p_y \cdot 0.5p_{y+1})\]

Solution to be discussed in lecture.
\[
= \left(1 - 0.95(1 - 0.5(0.06))\right) \left(1 - 0.90(1 - 0.5(0.12))\right)
\]

\[
= \frac{\ln 0.9879}{0.9879} = 0.018
\]

\[
M_{x+t} = -\frac{d}{dt} \ln p_x = \left. \frac{-1}{tp_x} \frac{1}{t} \right|_{p_x} = \frac{f_{Tx}(t)}{S_{Tx}(t)}
\]

\[
M_{x+t:y+t} = \frac{f_{Txy}(t)}{S_{Txy}(t)}
\]
Force of mortality of $T_{xy}$

Define the force of mortality (similar manner to any random variable):

$$
\mu_{x+t:y+t} = \frac{f_{T_{xy}}(t)}{1 - F_{T_{xy}}(t)} = \frac{f_{T_{xy}}(t)}{S_{T_{xy}}(t)} = \frac{f_{T_{xy}}(t)}{tP_{xy}}.
$$

- We can then write the density of $T_{xy}$ as

$$
f_{T_{xy}}(t) = tP_{xy} \cdot \mu_{x+t:y+t}.
$$

- In the case of independence, we have:

$$
\mu_{x+t:y+t} = \frac{tP_x \cdot tP_y (\mu_{x+t} + \mu_{y+t})}{tP_x \cdot tP_y} = \mu_{x+t} + \mu_{y+t}.
$$

- The force of mortality of the joint life status is the sum of the individuals’ force of mortality, when lives are independent.
The force of mortality for $T_{xy}$ is defined as

$$
\mu_{x+t:y+t} = \frac{f_{T_{xy}}(t)}{1 - F_{T_{xy}}(t)} = \frac{f_{T_{xy}}(t)}{S_{T_{xy}}(t)} = \frac{f_{T_{xy}}(t)}{f_x(t) + f_y(t) - f_{T_{xy}}(t)}
$$

Indeed we have the density of $T_{xy}$ expressed as

$$
T_{xy} + T_{xy} = T_x + T_y
$$

Check what this formula gives in the case of independence.
Consider an insurance under which the benefit of $1 is paid at the EOY of ending (failure) of status $u$.

Status $u$ could be any joint life or last survivor status e.g. $xy, \overline{xy}$. Then

- the time at which the benefit is paid: $K_u + 1$
- the present value (at issue) of the benefit: $Z = v^{K_u+1}$
- APV of benefits: $E[Z] = A_u = \sum_{k=0}^{\infty} v^{k+1} \cdot \Pr[K_u = k]$
- variance: $\text{Var}[Z] = 2A_u - (A_u)^2$
Insurance benefits - continuous

- Consider an insurance under which the benefit of $1 is paid immediately of ending (failure) of status $u$.
- Status $u$ could be any joint life or last survivor status e.g. $xy, \overline{xy}$. Then:
  - the time at which the benefit is paid: $T_u$
  - the present value (at issue) of the benefit: $Z = v^{T_u}$
  - APV of benefits: $E[Z] = \overline{A}_u = \int_0^\infty v^t \cdot t p_u \cdot \mu_{u+t} dt$
  - variance: $\text{Var}[Z] = 2 \overline{A}_u - (\overline{A}_u)^2$
Some illustrations

- For a **joint life status** \((xy)\), consider whole life insurance providing benefits at the first death:

\[
A_{xy} = \sum_{k=0}^{\infty} v^{k+1} \cdot k|q_{xy} = \sum_{k=0}^{\infty} v^{k+1} \cdot kP_{xy} \cdot q_{x+k:y+k}
\]

\[
\bar{A}_{xy} = \int_{0}^{\infty} v^t \cdot tP_{xy} \cdot \mu_{x+t:y+t} dt
\]

- For a **last-survivor status** \((\overline{xy})\), consider whole life insurance providing benefits upon the last death:

\[
A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \cdot k|q_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \cdot (k|q_{x} + k|q_{y} - k|q_{xy})
\]

\[
\bar{A}_{\overline{xy}} = \bar{A}_{x} + \bar{A}_{y} - \bar{A}_{xy}
\]

\[
\bar{A}_{xy} = \int_{0}^{\infty} v^t \left( tP_{x} \cdot \mu_{x+t} + tP_{y} \cdot \mu_{y+t} - tP_{xy} \cdot \mu_{x+t:y+t} \right) dt
\]
\[ A_{xy} = \sum_{k=0}^{\infty} v^{k+1} \cdot P_r(K_{xy} = k+1) \]

\[ k! q_x = k! q_x k \]

\[ k! q_{xy} = k! p_{xy} q_{x+k:y+k} \]

\[ k! q_{xy} = k! p_{xy} q_{x+k:y+k} \]

\[ A_{xy} = A_x + A_y - A_{xy} \]

\[ A_{xy} = A_x + A_y - A_{xy} \]
Useful relationships:

\[ A_{xy} + \bar{A}_{xy} = A_x + A_y \]

\[ \bar{A}_{xy} + \bar{A}_{xy} = \bar{A}_x + \bar{A}_y \]
Annuity benefits - discrete

Consider an $n$-year temporary life annuity-due on status $u$. Then

- the present value (at issue) of the benefit: $Y = \begin{cases} \ddot{a}_{K_u+1}, & K_u < n \\ \ddot{a}_{\infty}, & K_u \geq n \end{cases}$

- APV of benefits: $E[Y] = \ddot{a}_{u: \infty} = \sum_{k=0}^{n-1} \ddot{a}_{k+1} \cdot k \cdot q_u + \ddot{a}_{\infty} \cdot n \cdot p_u$

- variance: $\text{Var}[Y] = \frac{1}{d^2} \left[ 2A_{u: \infty} - (A_{u: \infty})^2 \right]$

- Other ways to write APV:

$$\ddot{a}_{u: \infty} = \sum_{k=0}^{n-1} v^k \cdot k \cdot p_u = \frac{1}{d} \left( 1 - A_{u: \infty} \right).$$
Consider an annuity for which the benefit of $1 is paid each year continuously for $\infty$ years so long as a status $u$ continues.

- Then
  
  - the present value (at issue) of the benefit: $Y = \bar{a}_{T_u}$
  
  - APV of benefits: $E[Y] = \bar{a}_u = \int_0^\infty \bar{a}_t \cdot i p_u \cdot \mu_{u+t} dt = \int_0^\infty v^t p_u dt$
  
  - variance: $\text{Var}[Y] = \frac{1}{\delta^2} \left[ 2 \bar{A}_u - (\bar{A}_u)^2 \right]$

- Note that the identity $\delta \bar{a}_{T_u} + v^{T_u} = 1$ provides the connection between insurances and annuities.
Some illustrations

- For joint life status \((xy)\), consider a whole life annuity providing benefits until the first death:

\[
\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \cdot kP_{xy} \quad \text{and} \quad \ddot{a}_{xy} = \int_{0}^{\infty} v^t \cdot tP_{xy} dt
\]

- For last survivor status \((\overline{xy})\), consider a whole life insurance providing benefits upon the last death:

\[
\ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k \cdot kP_{xy} \quad \text{and} \quad \ddot{a}_{xy} = \int_{0}^{\infty} v^t \cdot tP_{xy} dt
\]

- Useful relationships:

\[
\ddot{a}_{xy} + \ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y
\]

\[
\ddot{a}_{xy} + \ddot{a}_{\overline{xy}} = \ddot{a}_x + \ddot{a}_y
\]
Comparing benefits - annuities

<table>
<thead>
<tr>
<th>Type of life annuity</th>
<th>Single life $x$</th>
<th>Joint life status $xy$</th>
<th>Last survivor status $\overline{xy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole life a-due</td>
<td>$\ddot{a}_x$</td>
<td>$\ddot{a}_{xy}$</td>
<td>$\ddot{a}_{\overline{xy}}$</td>
</tr>
<tr>
<td>Whole life a-immediate</td>
<td>$a_x$</td>
<td>$a_{xy}$</td>
<td>$a_{\overline{xy}}$</td>
</tr>
<tr>
<td>Temporary life a-due</td>
<td>$\ddot{a}_{x:n}$</td>
<td>$\ddot{a}_{xy:n}$</td>
<td>$\ddot{a}_{\overline{xy}:n}$</td>
</tr>
<tr>
<td>Temporary life a-immediate</td>
<td>$a_{x:n}$</td>
<td>$a_{xy:n}$</td>
<td>$a_{\overline{xy}:n}$</td>
</tr>
<tr>
<td>Whole life a-continuous</td>
<td>$\overline{a}_x$</td>
<td>$\overline{a}_{xy}$</td>
<td>$\overline{a}_{\overline{xy}}$</td>
</tr>
<tr>
<td>Temporary life a-continuous</td>
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<td>$\overline{a}_{\overline{xy}:n}$</td>
</tr>
</tbody>
</table>
Comparing benefits - insurances

<table>
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<tr>
<th>Type of life insurance</th>
<th>Single life $x$</th>
<th>Joint life status $xy$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Whole life - discrete</td>
<td>$A_x$</td>
<td>$A_{xy}$</td>
<td>$A_{x\overline{y}}$</td>
</tr>
<tr>
<td>Whole life - continuous</td>
<td>$\overline{A}_x$</td>
<td>$\overline{A}_{xy}$</td>
<td>$\overline{A}_{x\overline{y}}$</td>
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<td>Term - discrete</td>
<td>$A_{1:x:n}$</td>
<td>$A_{1:xy:n}^{-}$</td>
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<td>Term - continuous</td>
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</tr>
<tr>
<td>Endowment - discrete</td>
<td>$A_{x:n}$</td>
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</tr>
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<td>Endowment - continuous</td>
<td>$\overline{A}_{x:n}$</td>
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</tr>
<tr>
<td>Pure endowment</td>
<td>$A_{x:1:n}$ or $nE_x$</td>
<td>$A_{xy:1:n}$ or $nE_{xy}$</td>
<td>$A_{xy:1:n}$ or $nE_{xy}$</td>
</tr>
</tbody>
</table>
\( (x, y, (T_x, T_y) \sim f_{T_x, T_y} \) \\

**Recall:** joint life status: \((xy)\) \(T_{xy} = \min(T_x, T_y)\) \\
last survivor status: \((\overline{xy})\) \(\overline{T}_{xy} = \max(\overline{T}_x, \overline{T}_y)\)

\[
P(T_{xy} > t) = P(T_x > t, T_y > t) = S_x(t, t)
\]

\[
\overline{t}P_{xy}
\]

\[
P(T_{\overline{xy}} \leq t) = P(T_x \leq t, T_y \leq t) = F_x(t, t)
\]

independence

\[
tP_{xy} = tP_x \cdot tP_y \quad tQ_{xy} = 1 - tP_{xy} = 1 - tP_x \cdot tP_y
\]

\[
= tQ_x tP_y + tP_x tQ_y + tQ_x tQ_y
\]
\[
T_x + T_y = T_{xy} + T_{xy}
\]
\[
t P_{xy} = t P_x + t P_y - t P_{xy}
\]
\[
t Q_{xy} = t Q_x + t Q_y - t Q_{xy}
\]
\[
\frac{f_{T_{xy}}}{f_{T_x}} (t) = t P_{xy} \frac{M_{x+t} \cdot y+t}{M_{x+t} \cdot y+t}
\]

If \( T_x, T_y \) are independent,
\[
\frac{f_{T_{x}, T_y}}{f_{T_x}} (t, s) = \frac{f_{T_x}}{f_{T_x}} (t) \cdot \frac{f_{T_y}}{f_{T_y}} (s)
\]
\[ F_{T_{xy}}(t) = P(T_{xy} \leq t) = 1 - P(T_x > t, T_y > t) \]

\[ = 1 - \frac{P(T_x > t) P(T_y > t)}{tP_x tP_y} \]

\[ 2F_{T_{xy}}(t) = \frac{1}{2t} \frac{\partial}{\partial t} \left( \frac{1}{tP_x tP_y} \right) = -tP_x \frac{\partial}{\partial t} tP_y - tP_y \frac{\partial}{\partial t} tP_x \]

\[ = -tP_x tP_y M_{xy} + tP_x tP_y \]

\[ = tP_x tP_y (M_{x+t} + M_{y+t}) \]

\[ = tP_{xy} (M_{x+t} + M_{y+t}) \]

\[ = tP_{xy} M_{x+t, y+t} \]
\begin{align*}
\bar{u} &= \text{status} \\
\vec{A}_u &= \mathbb{E}[U^{T_u}] \implies \int_0^\infty v \cdot f_{T_u}(t) \, dt \\
A_u &= \mathbb{E}[V^{K_u+1}] \implies \sum_{k=0}^{\infty} v^{K_u} q_u \\
\overline{A}_{xy} &= \mathbb{E}[V^{T_{xy}}] = \int_0^\infty v^t \cdot t \cdot p_{xy} (x+t,y+t) \, dt \\
\overline{A}_{xy} &= \overline{A}_x + \overline{A}_y - \overline{A}_{xy}
\end{align*}
Illustrative example 3

You are given:

- \( (45) \) and \( (65) \) have independent future lifetimes.
- Mortality for either life follows deMoivre’s law with \( \omega = 105 \).
- \( \delta = 5\% \)

Calculate \( \overline{A}_{45:65} \).
\[ f_{\frac{T_{45:65}}{t}}(t) = \frac{dF(t)}{dt}, \quad f(t) = \frac{1}{60} \]

\[ F_{\frac{T_{45:65}}{t}}(t) = \frac{1}{60} \left( \Pr(T_{45} \leq t) \Pr(T_{65} \leq t) \right) \]

\[ = \begin{cases} 
\frac{t}{60} \frac{t}{40}, & 0 \leq t \leq 40 \\
\frac{t}{40}, & 40 \leq t \leq 60 \\
0, & \text{else} 
\end{cases} \]
\[ f_{T_{45:65}}(t) = \begin{cases} \frac{t}{1200}, & 0 \leq t \leq 40 \\ \frac{1}{60}, & 40 < t \leq 60 \\ 0, & \text{else} \end{cases} \]

\[ E[T_{45:65}] = \int_0^{40} v^t \frac{t}{1200} \, dt + \int_{40}^{60} v^t \frac{1}{60} \, dt + \varphi \]

\[ v^t = e^{-0.05t} \]

\[ \int t \, dv^t = \int t \, e^{-0.05t} \, dt = \frac{T_{45:65}^t}{2} \left( e^{-0.05} - 1 \right) + C \]

\[ \bar{A}_{45:65} = \frac{1}{1200} \frac{T_{45:65}^t}{237.5477} + \frac{1}{60} \frac{T_{45:65}^t}{1.765964} \]

\[ = 0.2265141 \]
the A.P.V. of an insurance that pays $1 upon the last death.

\[ \bar{A}_{45:65} = \text{the A.P.V. of an insurance that pays $1 upon the first death} \]

\[ \bar{A}_{45} + \bar{A}_{65} - \bar{A}_{45:65} \approx 0.2265141 \]

\[ \int_{0}^{60} \frac{1}{60} v^t dt + \int_{0}^{40} \frac{1}{40} v^t dt > 0.2265741 \]
It also works for annuities, e.g. whole life annuity-due on status (u) 

\[ PV = \ddot{A}_{u+1} \]

\[ E[PV] = \ddot{A}_u = \sum_{k=0}^{\infty} \ddot{a}_{k+1} \cdot k! q_u \]

Current payment technique

\[ \ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k k p_{xy} \]

Last survivor \[ \ddot{a}_{xy} = \sum_{k=0}^{\infty} v^k k \overline{p}_{xy} = \ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy} \]
\[ \ddot{a} \rightarrow A \]

\[
E\left[ \ddot{A}_{k_{n+1}} \right] = E\left[ \frac{1 - V}{d} \right] \\
= 1 - E\left[ \frac{V^{k_{n+1}}}{d} \right]
\]

e.g.

\[
\ddot{A}_{x_{xy}} = \frac{1 - A_{x_{xy}}}{d} \Rightarrow A_{x_{xy}} = 1 - d \ddot{A}_{x_{xy}}
\]

\[
\ddot{A}_{\overline{x_{xy}}} = \frac{1 - A_{\overline{x_{xy}}}}{d} \Rightarrow A_{\overline{x_{xy}}} = 1 - d \ddot{A}_{\overline{x_{xy}}}
\]

Continuum also works:

\[ \ddot{A}_{x_{xy}} \neq \ddot{A}_{x} \ddot{A}_{y} \]

\[
A_{x_{xy}} = A_{x} + A_{y} - A_{\overline{x_{xy}}} \\
\ddot{A}_{x_{xy}} = \ddot{A}_{x} + \ddot{A}_{y} - \ddot{A}_{\overline{x_{xy}}}
\]
\( \bar{A}_{xy} = \) the APV of an annuity that pays $1 at the beginning of each year that both \((x)\) and \((y)\) are alive

\( \bar{\bar{A}}_{xy} = \) the APV of an annuity that pays $2 at the beginning of each year that at least one of \((x)\) or \((y)\) is alive
Pure endowment:

\[ nE_x = v^n p_x \]
\[ nE_y = v^n p_y \]
\[ nE_{xy} = v^n p_{xy} \neq nE_x \cdot nE_y \]
\[ (1+i)^n nE_x \cdot nE_y \]

\[ nE_{\overline{xy}} = nE_x + nE_y - nE_{xy} \] if independent.
Contingent functions

- It is possible to compute probabilities, insurances and annuities based on the failure of the status that is contingent on the order of the deaths of the members in the group, e.g. $(x)$ dies before $(y)$.

- These are called contingent functions.

- Consider the probability that $(x)$ fails before $(y)$ - assuming independence:

$$
\Pr[T_x < T_y] = \int_0^\infty t \cdot p_x \mu_{x+t} \cdot p_y dt
$$

- The actuarial symbol for this is $q_{1\infty}^{xy}$. It should be obvious this is the same as $\infty q_{xy}^2$. 
\( f_{xy}^2 = f_{xy} \quad \rightarrow \quad b + n f_{xy}^2 \neq n f_{xy}^2 \)

\( f_{xy} = f_{xy} \quad \rightarrow \quad \int_0^\infty p_{xy} m_{y+t} \, dt \)

\( \int_0^\infty q_{xy} t \, p_x \, m_{x+t} \, dt \)

\( n f_{xy}^2 + n \dot{f}_{xy} = \dot{n} p_{xy} - n \dot{f}_{xy} \) independent
In summary, we continue to

\[ \int_{\gamma} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial \Omega} \mathbf{F} \cdot d\mathbf{S} \]

So if \( \mathbf{F} = p \mathbf{x} + q \mathbf{y} \)

\[ \int_{\gamma} p \mathbf{x} + q \mathbf{y} \cdot d\mathbf{r} + \int_{\partial \Omega} p x d\mathbf{S} + q y d\mathbf{S} \]

\[
\begin{align*}
\int_{\gamma} & p x d\mathbf{S} + \int_{\partial \Omega} q y d\mathbf{S} \\
= & \int_{\gamma} p x d\mathbf{S} + \int_{\partial \Omega} q y d\mathbf{S} \\
\end{align*}
\]

- Remember
  - April 18, 2018
  - February 15,
- continued

- The probability that \( (x) \) dies before \( (y) \) and within \( n \) years is given by

\[
nq_{1}^{nxy} = \int_{0}^{n} t p_{xy} \mu_{x+t} dt.
\]

- Similarly, we have the probability that \( (y) \) dies before \( (x) \) and within \( n \) years:

\[
nq_{2}^{nxy} = \int_{0}^{n} t p_{xy} \mu_{y+t} dt.
\]

- It is easy to show that \( nq_{xy}^{1} + nq_{xy}^{1} = nq_{xy} \).

- One can similarly define and interpret the following: \( nq_{xy}^{2} \) and \( nq_{xy}^{2} \), and show that

\[
nq_{xy}^{2} + nq_{xy}^{2} = nq_{\overline{xy}}.
\]
\[ n \overrightarrow{Q}_1 = \frac{\int_0^t \overrightarrow{F} \times \overrightarrow{M} x \ dt}{\int_0^t f y + p x \ \overrightarrow{M} x + x \ dt} \]

\[ n \overrightarrow{Q}_2 = \frac{\int_0^t \overrightarrow{F} \times \overrightarrow{M} y \ dt}{\int_0^t f x + p y \ \overrightarrow{M} y + y \ dt} \]

\[ n \overrightarrow{Q}_3 = \frac{\int_0^t \overrightarrow{F} \times \overrightarrow{M} z \ dt}{\int_0^t f z + p z \ \overrightarrow{M} z + z \ dt} \]

\[ n \overrightarrow{Q}_4 = \frac{n q_x + n q_y - n q_{xy}}{n q_{xyz}} \]
\[ n^q_{x y z} = \int_0^t \int_0^n p_{x y z} M_{y+t} \, dt \]

\[ n^q_{x y z} = \int_0^n \int_0^t q_x t p_y t p_z M_{z+t} \, dt \]
\[ \overline{A}_{x \overline{y}} \]

Insurance pays at the moment \((y)\) dies provided \((y)\) is predeceased by \((x)\), predeceased \((y)\) dies after \((x)\) + 1

\[ \int_0^{\infty} e^{-t \lambda x} \cdot q_{x+t} p_y \cdot M_{y+t} \, dt \]

\[ \overline{A}_{x \overline{y}} = \overline{A}_y - \overline{A}_{x \overline{y}} \Rightarrow \overline{A}_{x \overline{y}} + \overline{A}_{x \overline{y}}^2 = \overline{A}_y \]
\[ PV = \begin{cases} \sqrt{T_y}, & T_y \geq T_x \\ 0, & T_y < T_x \end{cases} \]

\[ = \sqrt{T_y} \cdot I(T_y \geq T_x) \]

\[ E[PV] = E[\sqrt{T_y} \cdot I(T_y \geq T_x)] \rightarrow A_{xy}^2 \]

\[ + E[\sqrt{T_y} \cdot I(T_y \leq T_x)] \rightarrow A_{xy}^2 \text{ or } A_{xy}' \]

\[ E[\sqrt{T_y} \cdot 1] = A_y \]
Illustrative example 4

An insurance of $1 is payable at the moment of death of \((y)\) if predeceased by \((x)\), i.e. if \((y)\) dies after \((x)\). The actuarial present value (APV) of this insurance is denoted by \(\overline{A}_{xy}^2\). Assume \((x)\) and \((y)\) are independent.

1. Give an expression for the present value random variable for this insurance.

2. Show that

\[
\overline{A}_{xy}^2 = \overline{A}_y - \overline{A}_{xy}^1.
\]

3. Prove that

\[
\overline{A}_{xy}^2 = \int_0^\infty v^t \overline{A}_{y+t} \overline{iP}_{xy} \mu_x + t dt,
\]

and interpret this result.
Take the case of constant free \_{Stop here}:

\[ T_x \sim \text{exponential with } \mu_x = 0.2 \]

\[ T_y \sim \text{with } \mu_y = 0.1 \]

\[ \bar{A}_x = \frac{\mu_x}{\mu_x + \delta} \]

Derive, \[ \bar{A}_{xy}, \bar{a}_{xy}, \bar{A}_{xy}, \bar{a}_{xy} \]

\[ \bar{A}_{xy} = \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \]

\[ \bar{A}_{xy} = 1 - \delta \bar{a}_{xy} \]
Singh life / discrete given constant force.

\[ A_x = \sum_{k=0}^{\infty} \nu (V_{x+k})^{k+1} q_x \]

\[ \ddot{a}_x = \sum_{k=0}^{\infty} \nu k p_x \]

\[ \dddot{a}_x = \sum_{k=0}^{\infty} \nu k^2 p_x \]

\[ A_{xy} = \sum_{k=0}^{\infty} e^{-\delta k} (1-e^{-M_x k}) \]

\[ A_{\bar{xy}} = \sum_{k=0}^{\infty} e^{-\delta k} \]

\[ A_x = 1 - d \ddot{a}_x \]

Exam 2: STOP HERE
Reversionary annuity
pay commences upon death of the other.

\( \bar{A}_{y|\delta x} = \int_0^\infty v^t p_{xy} \mu_{yt+t} \bar{A}_{x+t} \, dt \)

\( \text{CPT} \quad \quad = \int_0^\infty v^t p_x (1-t p_y) \, dt = \bar{A}_x - \bar{A}_{xy} \)
Reversionary annuities

A reversionary annuity is an annuity which commences upon the failure of a given status \((u)\) if a second status \((v)\) is then alive, and continues thereafter so long as status \((v)\) remains alive.

- Consider the simplest form: an annuity of $1 per year payable continuously to a life now aged \(x\), commencing at the moment of death of \((y)\) - briefly annuity to \((x)\) after \((y)\).

- APV for this reversionary annuity:

\[
\bar{a}_{y|x} = \int_{0}^{\infty} v^t t p_{xy} \mu_y + t \bar{a}_{x+t} dt.
\]

- One can show the more intuitive formula (using current payment technique):

\[
\bar{a}_{y|x} = \int_{0}^{\infty} v^t t p_{x} (1 - t p_y) dt = \bar{a}_{x} - \bar{a}_{xy}.
\]
Reversionary annuities

Present value random variable

- For the reversionary annuity considered in the previous slides, one can also write the present-value random variable at issue as:

$$Z = \begin{cases} T_y \bar{a}_{T_x-T_y}, & T_y \leq T_x \\ 0, & T_y > T_x \end{cases}$$

- By taking the expectation of $Z$, we clearly have $\bar{a}_{y\mid x} = \bar{a}_x - \bar{a}_{xy}$. 

Can you explain the last line?
Reversionary annuities - discrete

- In general, an annuity to any status \((u)\) after status \((v)\) is

\[ a_{v\mid u} = a_u - a_{uv} \]

where \(a\) is any annuity which takes discrete, continuous, or payable \(m\) times a year.

- Consider the discrete form of reversionary annuity: $1 per year payable to a life now aged \(x\), commencing at the EOY of death of \((y)\).

- APV for this reversionary annuity:

\[ a_{y\mid x} = \sum_{k=1}^{\infty} v^k k p_x (1 - k p_y) = a_x - a_{xy}. \]

- If \((v)\) is the term-certain \((\overline{n})\) and \((u)\) is the single life \((x)\), then

\[ a_{\overline{n}\mid x} = a_x - a_{x: \overline{n}} \]

which is indeed a single-life deferred annuity.
\[ a_{y|xz} = a_{xz} - a_{xyz} \]

You pay annually starting upon death of \((y)\) until the first death of \((x), (z)\).

\[ a_{y|xz} = a_{xz} - a_{y:xz} \]

Rewritten:

\[ tP_{xy} = tP_{xy}^{00} \]

\[ tP_{xy} = 1 - tP_{xy}^{03} = tP_{xy}^{01} + tP_{xy}^{02} + tP_{xy}^{00} \]
Back to multiple state framework

Translating the probabilities/forces earlier defined, the following should now be straightforward to verify:

- \( tP_{xy} = tP_{xy}^{00} \)
- \( tQ_{xy} = tP_{xy}^{01} + tP_{xy}^{02} + tP_{xy}^{03} \)
- \( tQ_{x\overline{y}} = tP_{xy}^{00} + tP_{xy}^{01} + tP_{xy}^{02} \)
- \( tQ_{\overline{x}y} = tP_{xy}^{03} \)
- \( tQ_{\overline{x}\overline{y}} = tP_{xy}^{03} \)
- \( tQ_{xy}^1 = \int_0^t sP_{xy} \mu_{x+s:y+s} ds \)
- \( tQ_{xy}^2 = \int_0^t sP_{xy} \mu_{x+s} ds \)
Annuities

In terms of the annuity functions, the following should also be straightforward to verify:

- $\bar{a}_{xy} = \bar{a}_{xy} = \int_0^\infty e^{-\delta t} tP_{xy}^{00} dt$

- $\bar{a}_{xy} = \bar{a}_{xy}^{00} + \bar{a}_{xy}^{01} + \bar{a}_{xy}^{02} = \int_0^\infty e^{-\delta t} (tP_{xy}^{00} + tP_{xy}^{01} + tP_{xy}^{02}) dt$

- $\bar{a}_{x|y} = \bar{a}_{xy}^{02} = \int_0^\infty e^{-\delta t} tP_{xy}^{02} dt$

The following also holds true (easy to verify):

- $\bar{a}_{xy} = \bar{a}_x + \bar{a}_y - \bar{a}_{xy}$

- $\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$
In terms of insurance functions, the following should also be straightforward to verify:

\[ \overline{A}_{xy} = \int_0^\infty e^{-\delta t} tP_{xy}^{00} (\mu_x + t: y + t) dt \]

\[ \overline{A}_{xy} = \int_0^\infty e^{-\delta t} (tP_{xy}^{13} \mu_x + tP_{xy}^{23} \mu_y) dt \]

\[ \overline{A}_{xy}^1 = \int_0^\infty e^{-\delta t} tP_{xy}^{00} \mu_x + tP_{xy}^{02} \mu_y + t dt \]

\[ \overline{A}_{xy}^2 = \int_0^\infty e^{-\delta t} tP_{xy}^{01} \mu_x + tP_{xy}^{13} \mu_y + t dt \]

The following also holds true (easy to verify):

\[ \overline{A}_{xy} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy} \quad \text{and} \quad \overline{a}_{xy} = \frac{1}{\delta} (1 - \overline{A}_{xy}) \]

\[ \overline{A}_{xy}^1 + \overline{A}_{xy}^2 = \overline{A}_x \]
The case of independence

\[ \begin{align*}
    \mu_m x + t &+ t \\
    \mu_f y + t &+ t \\
    x \text{ alive} &\quad y \text{ alive} \\
    (0) &\quad (1)
\end{align*} \]

\[ \begin{align*}
    \mu_m x + t &
    \mu_f y + t \\
    x \text{ dead} &\quad y \text{ alive} \\
    (2) &\quad (3)
\end{align*} \]
Illustrative example 5

Suppose that the future lifetimes, $T_x$ and $T_y$, of a husband and wife, respectively are independent and each is uniformly distributed on $[0, 50]$. Assume $\delta = 5\%$.

1. A special insurance pays $1 upon the death of the husband, provided that he dies first. Calculate the actuarial present value for this insurance and the variance of the present value.

2. An insurance pays $1 at the moment of the husband’s death if he dies first and $2 if he dies after his wife. Calculate the APV of the benefit for this insurance.

3. An insurance pays $1 at the moment of the husband’s death if he dies first and $2 at the moment of the wife’s death if she dies after her husband. Calculate the APV of the benefit for this insurance.
1. \[ \text{APV} = \int_0^{50} v^t \cdot \bar{P}_x M_{xt} \cdot \bar{P}_y \, dt \]
   \[ = \int_0^{50} e^{-0.05t} \cdot \frac{1}{50} \cdot (1 - \frac{t}{50}) \, dt \]
   \[ = 1.253133 \]

   Variance = APV \cdot 28 - (\text{APV})^2

   \[ = \int_0^{50} e^{-0.10t} \cdot \frac{1}{50} \cdot (1 - \frac{t}{50}) \, dt - (1.253133)^2 \]
   \[ = 0.961928 \]

2. \[ \text{APV} = 1 - \int_0^{50} v^t \cdot \bar{P}_x M_{xt} \cdot \bar{P}_y \, dt + 2 \int_0^{50} v^t \cdot \bar{P}_x M_{xt} \cdot \bar{P}_y \, dt \]
   \[ = 2 \int_0^{50} v^t \cdot \bar{P}_x M_{xt} \cdot \bar{P}_y \, dt - \text{APV}(\text{@}1) \]
\[ T_x \sim U(0, 50) \]
\[ T_y \sim U(0, 50) \]

\[ e^{-\delta t} \frac{1}{50} \]

\[ = 0.04811984 \]

\[ \text{APV} = \overbrace{\text{APV}(\theta, t)} + 2 \int_{0}^{\infty} v^+ t y_{\theta} \mu_{y+i+t} + \bar{y} \text{d}t}^{\text{APV}(\theta, t)} \]

\[ = 3 \times (0.2531536) = 0.7594008 \]
Illustrative example 6

For a husband and wife with ages $x$ and $y$, respectively, you are given:

- $\mu_{x+t} = 0.02$ for all $t > 0$
- $\mu_{y+t} = 0.01$ for all $t > 0$
- $\delta = 0.04$

1. Calculate $\bar{a}_{xy:20}$ and $\bar{a}_{xy:20}$.

2. Rewrite this problem in a multiple state framework and solve (1) within this framework.
\[ \mathbf{A}_{xy: 20} = \int_0^{20} v^t + p_{xy} \, dt \]

\[ e^{-0.04t} + p_x + p_y = -\mu_x t - \mu_y t \]

\[ e^{-0.02t} - \mu_x t - \mu_y t \]

\[ = (10.7629) \]

\[ \mathbf{a}_{xy: 20} = \mathbf{a}_{x: 20} + \mathbf{a}_{y: 20} - \mathbf{A}_{xy: 20} \]

\[ \int_0^{20} e^{-0.04t} + p_x \, dt \]

\[ \int_0^{20} e^{-0.02t} + p_y \, dt \]

\[ \int_0^{20} e^{-0.01t} \]

\[ = 13.52627 \]
For $(x)$ and $(y)$ with independent future lifetimes, you are given:

- $\bar{a}_x = 10.06$
- $\bar{a}_y = 11.95$
- $\bar{a}_{xy} = 12.59$
- $\bar{A}_{1}^{xy} = 0.09$
- $\delta = 0.07$

Calculate $\bar{A}_{xy}^{1}$. 

SOA
dom in practice on #9

similar on 12 test #9
The model with a common shock

\[ \mu_{02} x + t : y + t = y + t \mu_{13} x + t : y + t = y + t \mu_{03} x + t : y + t = y + t \mu_{01} x + t : y + t = y + t \mu_{23} y + t = y + t \]

\( x \) alive
\( y \) alive
(0)

\( x \) dead
\( y \) alive
(2)

\( x \) alive
\( y \) dead
(1)

\( x \) dead
\( y \) dead
(3)

\( \sigma = 0.05 \)
Illustrative example 8: SOA Spring 2014 Question # 7

The joint mortality of two lines \((x)\) and \((y)\) is being modeled as a multiple state model with a common shock (see diagram in the previous page).

You are given:

- \(\mu^{01} = 0.010\)
- \(\mu^{02} = 0.030\)
- \(\mu^{03} = 0.005\)
- \(\delta = 0.05\)

A special joint whole life insurance pays \(1000\) at the moment of simultaneous death, if that occurs, and zero otherwise.

Calculate actuarial present value of this insurance.
\[ 1000 \times \int_0^{\infty} e^{-0.05t} \cdot 0.03 \cdot dt \\
= 1000 \cdot \left( \frac{0.005}{0.095} \right) = \frac{52.63158}{0.005} \]