3.5C Rational Functions and Asymptotes

A. Definition of a Rational Function

\( f(x) \) is said to be a rational function if \( f(x) = \frac{g(x)}{h(x)} \), where \( g \) and \( h \) are polynomial functions. That is, rational functions are fractions with polynomials in the numerator and denominator.

B. Asymptotes/Holes

Holes are what they sound like:

\[ \bigcirc \]

is a hole

Rational functions may have holes or asymptotes (or both!).

Asymptote Types:

1. vertical
2. horizontal
3. oblique ("slanted-line")
4. curvilinear (asymptote is a curve!)

We will now discuss how to find all of these things.
C. Finding Vertical Asymptotes and Holes

Factors in the denominator cause vertical asymptotes and/or holes.

To find them:

1. Factor the denominator (and numerator, if possible).
2. Cancel common factors.
3. Denominator factors that cancel completely give rise to holes. Those that don’t give rise to vertical asymptotes.

D. Examples

Example 1: Find the vertical asymptotes/holes for \( \frac{f(x)}{g(x)} \) where \( f(x) = \frac{(3x+1)(x-7)(x+4)}{(x-7)^2(x+4)} \).

Solution

Canceling common factors: \( f(x) = \frac{3x+1}{x-7}, \quad x \neq -4 \)

\( x + 4 \) factor cancels completely \( \Rightarrow \) hole at \( x = -4 \)

\( x - 7 \) factor not completely canceled \( \Rightarrow \) vertical asymptote with equation \( x = 7 \)

Example 2: Find the vertical asymptotes/holes for \( \frac{f(x)}{g(x)} \) where \( f(x) = \frac{2x^2-5x-12}{x^3-5x+4} \).

Solution

Factor: \( f(x) = \frac{(x-4)(2x+3)}{(x-4)(x+1)} \)
Cancel: \( \tilde{g}(x) = \frac{2x^3 + 3}{x - 1} \), \( x \neq 4 \)

Ans

<table>
<thead>
<tr>
<th>Hole at ( x = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical Asymptote with equation ( x = 1 )</td>
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E. Finding Horizontal, Oblique, Curvilinear Asymptotes

Suppose \( \tilde{g}(x) = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} \)

If

1. degree top < degree bottom: horizontal asymptote with equation \( y = 0 \)

2. degree top = degree bottom: horizontal asymptote with equation \( y = \frac{a_n}{b_m} \)

3. degree top > degree bottom: oblique or curvilinear asymptotes

To find them: Long divide and throw away remainder

F. Examples

Example 1: Find the horizontal, oblique, or curvilinear asymptote for \( f \) where \( f(x) = \frac{6x^4 - x + 2}{7x^4 + 2x^3 - 1} \).

Solution

degree top = 4 degree bottom = 5. Since 4 < 5, we have

Ans horizontal asymptote with equation \( y = 0 \)
Example 2: Find the horizontal, oblique, or curvilinear asymptote for \( f(x) = \frac{6x^3 - 2x^2 + 1}{2x^3 + 5} \).

Solution

Degree top = 3, degree bottom = 3.

Since 3 = 3, we have a horizontal asymptote of \( y = \frac{6}{2} = 3 \). Thus

\[ \text{Ans} \] horizontal asymptote with equation \( y = 3 \)

Example 3: Find the horizontal, oblique, or curvilinear asymptote for \( f(x) = \frac{2x^3 - 3}{x^2 - 1} \).

Solution

Degree top = 3, degree bottom = 2.

Since 3 > 2, we have an oblique or curvilinear asymptote. Now long divide:

\[
x^2 + 0x - 1 \left[ \frac{2x}{2x^3 + 0x^2 + 0x - 3} \right] \\
\quad = \frac{-2x^3 + 0x^2 - 2x}{2x - 3}
\]

Since \( \frac{2x^3 - 3}{x^2 - 1} = 2x + \frac{2x - 3}{x^2 - 1} \), we have that

\[ \text{Ans} \] \( y = 2x \) defines a line, and is the equation for the oblique asymptote
Example 4: Find the horizontal, oblique, or curvilinear asymptote for $f(x)$ where

$$f(x) = \frac{3x^5 - x^4 + 2x^2 + x + 1}{x^2 + 1}.$$

Solution

degree top = 5 degree bottom = 2.

Since $5 > 2$, we have an oblique or curvilinear asymptote. Now long divide:

$$\begin{align*}
3x^3 - x^2 - 3x + 3 \\
x^2 + 0x + 1 & \overline{3x^5 - x^4 + 0x^3 + 2x^2 + x + 1} \\
& - (3x^5 + 0x^4 + 3x^3) \\
& - x^4 - 3x^3 + 2x^2 \\
& - (-x^4 + 0x^3 - x^2) \\
& - 3x^3 + 3x^2 + x \\
& - (-3x^3 + 0x^2 - 3x) \\
& 3x^2 + 4x + 1 \\
& - (3x^2 + 0x + 3) \\
& 4x - 2
\end{align*}$$

Since $\frac{3x^5 - x^4 + 2x^2 + x + 1}{x^2 + 1} = 3x^3 - x^2 - 3x + 3 + \frac{4x - 2}{x^2 + 1}$, we have that

Ans $y = 3x^3 - x^2 - 3x + 3$ defines a curvilinear asymptote
G. Asymptote Discussion for Functions

1. As the graph of a function approaches a **vertical asymptote**, it shoots up or down toward $\pm \infty$.

2. Graphs approach **horizontal, oblique, and curvilinear asymptotes** as $x \to -\infty$ or $x \to \infty$.

3. Graphs of functions **never** cross vertical asymptotes, but **may** cross other asymptote types.