7.2C Absolute Value and Roots

A. Absolute Value Discussion

Recall that absolute value $|\cdot|$ means distance from the origin.

We think of absolute value of numbers as “make it positive”, but of course that doesn’t work for variables. (See Sections 2.3 and 2.7)

Recall that $|-3| = 3$ and $|4| = 4$ etc.

We know consider a third interpretation.

Notice the following:

$$|6| = 6$$
$$|3| = 3$$
$$|0| = 0$$

If the number inside is positive or zero, the absolute value does nothing.

Thus $|x| = x$; if $x \geq 0$.

Notice the following:

$$|-3| = 3$$
$$|-4| = 4$$

In this case, the sign changes.

**Question:** How can we change from $-3$ to $3$ or $-4$ to $4$ without using absolute value signs?
Answer: Multiply by $-1$

Thus notice that:

\[ | -3| \text{ is the same as } -(3) \]
\[ | -4| \text{ is the same as } -(4) \]

Thus, $|x| = -x$; if $x < 0$.

B. Definition of Absolute Value: Three Forms

1. For **numbers** only, “make it positive”

2. **True Definition:** distance from the origin

   This is the correct definition, and works for numbers or variables. This is needed for **equations** or **inequalities**

3. **Piecewise Definition:**

   \[
   |x| = \begin{cases} 
   x \text{; if } x \geq 0 \\
   -x \text{; if } x < 0 
   \end{cases}
   \]

C. Comments on the Piecewise Definition

1. The piecewise definition is the “formal definition” in terms of an algebraic formula.

2. The piecewise definition does **not** mean that $|x| = \pm x$ or some such nonsense. There is only one **answer** to $|x|$; however, the answer we choose **depends** on what’s inside.
3. When an object has more than one “formula”, and the expression you choose depends on some conditions, we say that the object is **piecewise defined**.

4. See MTH103 for more on piecewise definitions.

**D. Use of the Piecewise Definition of $|x|$ in Examples**

**Example 1:** Find $|7 - \sqrt{3}|$ exactly

**Solution**

$7 - \sqrt{3} \geq 0$; so we use $|x| = x$ in this case

Thus $|7 - \sqrt{3}| = 7 - \sqrt{3}$

Ans $7 - \sqrt{3}$

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**Example 2:** Find $|1 - \sqrt{17}|$ exactly

**Solution**

$1 - \sqrt{17} < 0$; so we use $|x| = -x$ in this case

Thus $|1 - \sqrt{17}| = -(1 - \sqrt{17})$

Ans $-1 + \sqrt{17}$
Example 3: Find \( |\sqrt{3} - \sqrt{10}| \) exactly

Solution

\[ \sqrt{3} - \sqrt{10} < 0; \text{ so we use } |x| = -x \text{ in this case} \]

Thus \( |\sqrt{3} - \sqrt{10}| = - (\sqrt{3} - \sqrt{10}) \)

Ans \[ \boxed{-\sqrt{3} + \sqrt{10}} \]

E. Roots and Powers

1. \((\sqrt[n]{x})^n = x\)

This is by definition of the \(n\)th root!

Thus “Root First, Then Power” \(\implies\) Cancel!

\((\sqrt[n]{x})^2 = x\)

\((\sqrt[n]{x})^3 = x\), etc.

2. The problem with \(\sqrt[n]{x^n}\)

We know that this is not the same situation.

Recall that we are only allowed to move powers inside if \(x\) is not simultaneously negative with \(n\) even.
Consider $\sqrt{x^2}$:

\[
\sqrt{4^2} = \sqrt{16} = 4
\]

\[
\sqrt{0^2} = \sqrt{0} = 0
\]

\[
\sqrt{(-3)^2} = \sqrt{9} = 3 \quad \text{not } -3!
\]

We see that, in fact, $\sqrt{x^2} = |x|$, since the answer is always positive (or zero)

We have a similar situation for all even roots:

\[
\sqrt{x^2} = |x|
\]

\[
\sqrt{x^4} = |x|
\]

\[
\sqrt{x^6} = |x|
\]

Since we don’t have any problem with odd roots, they just cancel:

\[
\sqrt[3]{x^3} = x
\]

\[
\sqrt[5]{x^5} = x
\]

Hence, we get another piecewise definition, depending on whether the index is even or odd:

\[
\sqrt[n]{x^n} = \begin{cases} x; & \text{if } n \text{ is odd} \\ |x|; & \text{if } n \text{ is even} \end{cases}
\]

Thus “Power First, Then Root” $\implies$ cancel only if the index is odd; otherwise absolute value!
F. Examples

Example 1: Find $\sqrt[3]{(7 - \sqrt{3})^3}$ exactly

Solution

Since the index is odd, we use $\sqrt[n]{x^n} = x$ in this case

Thus $\sqrt[3]{(7 - \sqrt{3})^3} = 7 - \sqrt{3}$

Ans $7 - \sqrt{3}$

Example 2: Find $\sqrt[4]{(10 - \sqrt{5})^4}$ exactly

Solution

Since the index is even, we use $\sqrt[n]{x^n} = |x|$ in this case

Thus $\sqrt[4]{(10 - \sqrt{5})^4} = |10 - \sqrt{5}|$

Now $10 - \sqrt{5} \geq 0$; so we use $|x| = x$ in this case

Thus $|10 - \sqrt{5}| = 10 - \sqrt{5}$

Ans $10 - \sqrt{5}$
Here’s where it gets interesting!

**Example 3:** Find \(\sqrt[6]{1 - \sqrt{7}}\) exactly

**Solution**

Since the index is **even**, we use \(\sqrt[n]{x^n} = |x|\) in **this** case

Thus \(\sqrt[6]{(1 - \sqrt{7})^6} = |1 - \sqrt{7}|\)

Now \(1 - \sqrt{7} < 0\); so we use \(|x| = -x\) in **this** case

Thus \(|1 - \sqrt{7}| = -(1 - \sqrt{7})\)

Ans \(-1 + \sqrt{7}\)

**Example 4:** Find \(\sqrt{(\sqrt[3]{6} - \sqrt[3]{13})^2}\) exactly

**Solution**

Since the index is **even**, we use \(\sqrt[n]{x^n} = |x|\) in **this** case

Thus \(\sqrt{(\sqrt[3]{6} - \sqrt[3]{13})^2} = |\sqrt[3]{6} - \sqrt[3]{13}|\)

Now \(\sqrt[3]{6} - \sqrt[3]{13} < 0\); so we use \(|x| = -x\) in **this** case

Thus \(|\sqrt[3]{6} - \sqrt[3]{13}| = -(\sqrt[3]{6} - \sqrt[3]{13})\)

Ans \(-\sqrt[3]{6} + \sqrt[3]{13}\)
G. Summary of Formulas

1. $\sqrt[n]{x} = x^{\frac{1}{n}}$ UNLESS index is even with $x$ possibly negative

2. $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^{m}$ UNLESS index is even with $x$ possibly negative

3. $\left(\sqrt[n]{x}\right)^{m} = \sqrt[n]{x^{m}}$ UNLESS index is even with $x$ possibly negative

4. Piecewise Definition of $|x|$: 

\[
|x| = \begin{cases} 
  x; & \text{if } x \geq 0 \\
  -x; & \text{if } x < 0 
\end{cases}
\]

5. $\sqrt[n]{x^n} = x$ “Root First, Then Power” $\implies$ CANCEL

6. Piecewise Definition of $\sqrt[n]{x^n}$: 

\[
\sqrt[n]{x^n} = \begin{cases} 
  x; & \text{if } n \text{ is odd} \\
  |x|; & \text{if } n \text{ is even} 
\end{cases}
\]