2.8 Distance Formula, Circles, Midpoint Formula

A. Distance Formula

We seek a formula for the distance between two points:

By the Pythagorean Theorem, \[d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2\]

Since distance is positive, we have:

Distance Formula: \[d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\]
B. Example

Find the distance between \((-1, 2)\) and \((3, -4)\)

Solution

Use the distance formula:

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
= \sqrt{(3 - (-1))^2 + (4 - 2)^2}
\]

\[
= \sqrt{4^2 + (-6)^2}
\]

\[
= \sqrt{16 + 36}
\]

\[
= \sqrt{52}
\]

Ans \(2\sqrt{13}\)

C. Circles

A circle is the set of points a fixed distance \(r\) from a center \((a, b)\):

![Diagram of a circle with center \((a, b)\) and radius \(r\)]

By the distance formula, \(r = \sqrt{(x - a)^2 + (y - b)^2}\)
Eliminating the radical, we get:

**Equation of Circle in Standard Form:**

\[(x - a)^2 + (y - b)^2 = r^2\]

**Note:** \(r\) is called the **radius** of the circle.

### D. Examples

#### Example 1:  
Find the equation of a circle with center \((2, -1)\) and radius 4.

**Solution**

The equation of a circle in standard form: \((x - a)^2 + (y - b)^2 = r^2\)

Thus, we have: \((x - 2)^2 + (y + 1)^2 = 4^2\)

**Ans** \((x - 2)^2 + (y + 1)^2 = 16\)

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#### Example 2:  
Given a circle \(x^2 + (y - 3)^2 = 5\), find the center and radius.

**Solution**

Since the equation of a circle in standard form is \((x - a)^2 + (y - b)^2 = r^2\), we have

**Ans**

- center: \((0, 3)\)
- radius: \(\sqrt{5}\)
E. Putting the Equation of a Circle in Standard Form

Sometimes the equation of a circle is not in standard form. To put it in standard form, we complete the square in both $x$ and $y$. In standard form, it is easy to identify the center and radius of the circle.

**Example 1:** Put the equation of the circle $x^2 + y^2 - 6x + 2y = 15$ into standard form.

**Solution**

$$x^2 + y^2 - 6x + 2y = 15$$

$$(x^2 - 6x) + (y^2 + 2y) = 15$$

$$[(x^2 - 6x + 9) - 9] + [y^2 + 2y + 1] - 1 = 15$$

$$(x - 3)^2 - 9 + (y + 1)^2 - 1 = 15$$

**Ans** \[(x - 3)^2 + (y + 1)^2 = 25\]

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**Example 2:** Put the equation of the circle $2x^2 + 2y^2 - 10x - 12y = 7$ into standard form.

**Solution**

$$2x^2 + 2y^2 - 10x - 12y = 7$$

$$(2x^2 - 10x) + (2y^2 - 12y) = 7$$

$$2(x^2 - 5x) + 2(y^2 - 6y) = 7$$

$$\left[2 \left( x^2 - 5x + \frac{25}{4} \right) - \frac{25}{2} \right] + \left[ 2(y^2 - 6y + 9) - 18 \right] = 7$$
\[ 2 \left( x - \frac{5}{2} \right)^2 + 2(y - 3)^2 = \frac{61}{2} + 7 \]

Ans \[ \left( x - \frac{5}{2} \right)^2 + (y - 3)^2 = \frac{75}{4} \]

F. Graphing Circles

To graph a circle:

1. Put the equation in standard form.
2. Find the center and radius.
3. Find the \( x \) and \( y \) intercepts.
4. Plot the \( x \) and \( y \) intercepts.
   Going any direction from the center by a radius amount reaches the circle.
5. Connect the dots.
Example: Find the center, radius, $x$ and $y$ intercepts of the circle, where $x^2 + y^2 - 2x + 8y = -5$. Then graph the circle.

Solution

First, put the circle in standard form:

\[
x^2 + y^2 - 2x + 8y = -5
\]

\[
(x^2 - 2x) + (y^2 + 8y) = -5
\]

\[
(x^2 - 2x + 1) - 1 + (y^2 + 8y + 16) - 16 = -5
\]

\[
(x - 1)^2 + (y + 4)^2 - 17 = -5
\]

\[
(x - 1)^2 + (y + 4)^2 = 12
\]

center: \((1, -4)\)

radius: \(\sqrt{12} = 2\sqrt{3}\)

$x$-intercepts: set $y = 0$:

\[
(x - 1)^2 + (0 + 4)^2 = 12
\]

\[
(x - 1)^2 + 16 = 12
\]

\[
(x - 1)^2 = -4
\]

\[
x - 1 = \pm \sqrt{-4}
\]

Thus there are no $x$-intercepts.
$y$-intercepts: set $x = 0$:

\[
(0 - 1)^2 + (y + 4)^2 = 12 \\
1 + (y + 4)^2 = 12 \\
(y + 4)^2 = 11 \\
y + 4 = \pm \sqrt{11} \\
y = -4 \pm \sqrt{11}
\]

Thus the $y$-intercepts are $-4 \pm \sqrt{11}$.

Graph:
G. Midpoint Formula

The **midpoint** between two points is the point on the line halfway between them.

Midpoint Formula: \[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

“average the \(x\)-coordinates and average the \(y\)-coordinates”

**Example:** Find the midpoint between \((-1, 2)\) and \((3, -4)\)

**Solution**

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{-1 + 3}{2}, \frac{2 + (-4)}{2} \right) = \left( \frac{2}{2}, \frac{-2}{2} \right)
\]

**Ans** \((1, -1)\)