Do any 7 of the following 10 exercises of your choice. Write up your solutions neatly in your own handwriting! Many problems are from the book, which is available online here: http://www-elec.inaoep.mx/~rogerio/FourierAnalysisUno.pdf. Students who are interested in understanding the material more completely are encouraged to solve problems 1, 2, 3, 4, 9, and 10 as part of their homework.

1. Do problem 1 on page 61 of Folland. For part (c) you should look up and apply the Dominated Convergence Theorem in order to make your solution rigorous. This problem shows that a function’s Fourier series may not converge uniformly to the function on \([-\pi, \pi]\) even if it is \(2\pi\) periodic, piecewise smooth, and has only one discontinuity! The upshot: continuity is both a necessary and sufficient condition for the uniform convergence of an arbitrary \(2\pi\) periodic and piecewise smooth function’s Fourier series to the function everywhere on \([-\pi, \pi]\) (see Theorem 2.5 on page 41 of Folland for sufficiency).

2. Recall the we defined the class \(C^k\) of functions \(f : [-\pi, \pi] \rightarrow \mathbb{C}\) to be those with a Riemann integrable \(k^{th}\) derivative on \([-\pi, \pi]\). We are also assuming periodicity so that all lesser derivatives of \(f\), \(f^{(l)}\), are continuous with \(f^{(l)}(\pi) = f^{(l)}(-\pi)\) for all \(l = 0, \ldots, k - 1\). For any \(f \in C^k\) we then showed that:

   (a) **THM 1/28 (A):** \(c_n = c_n^{(k)}/(in)^k\) for all \(n \neq 0\) (where \(c_n^{(l)}\) is the \(n^{th}\) Fourier series coefficient of \(f^{(l)}\), with \(c_n = c_n^{(0)}\)).

   (b) **THM 1/26:** If \(f \in C^k\) with \(k \geq 2\), then \(f(\theta) = \lim_{N \rightarrow \infty} \sum_{n=-N}^{N} c_n e^{in\theta}\) for all \(\theta \in [-\pi, \pi]\) (pointwise). If fact, we proved something a bit more general than this...but this will do here.

   Use these two facts to prove that, for every \(f \in C^k\) with \(k \geq 2\), there exists a constant \(B \in \mathbb{R}^+\) (which only depends on \(f^{(k)}\)) such that

   \[
   \left| f(\theta) - \sum_{n=-N}^{N} c_n e^{in\theta} \right| \leq \frac{B}{N^{k-1}}
   \]

   holds for all \(N \in \mathbb{Z}^+\) and \(\theta \in [-\pi, \pi]\). This proves that convergence of the Fourier series is both uniform as soon as \(f\) is smooth enough, and also generally faster the smoother \(f\) is.

3. The theorem you proved for #2 is useful in applications. Let’s try to understand it a little better with an example: Consider the function \(f(x) = \cos(100x)\). Note that \(f \in C^k\) for any \(k \in \mathbb{Z}^+\) you like.

   (a) Look back at your proof of Theorem 2 and figure out how large \(B\) is for \(f(x) = \cos(100x)\) when \(k = 13\). How does this \(B\) vary with \(k\) in general?

   (b) How large does the error bound in equation 1 tell you have to take \(N\) before you can be sure that your error will always be less than \(.001\) for all \(\omega \in [-\pi, \pi]\) when using \(k = 13\) for \(f(x) = \cos(100x)\)?

   (c) How large will your actual error be once you pick any \(N \geq 100\) for \(f(x) = \cos(100x)\)?

   (d) How large does your error bound in equation 1 tell you have to take \(N\) before you can be sure that your error will always be less than \(10^{-16}\) for all \(\omega \in [-\pi, \pi]\) as you let \(k \rightarrow \infty\) for \(f(x) = \cos(100x)\)?

4. Suppose you know that a function \(f : [-\pi, \pi] \rightarrow \mathbb{C}\) you want to learn about is composed of exactly one frequency component. That is, that \(f(x) = Ae^{i\omega x}\) for unknown parameters \(A \in \mathbb{C}\) and \(\omega \in \mathbb{Z} \cap [-127627, 127627]\). Use **THM 2/4** from class (see below) together with the Chinese Remainder Theorem in order to show that you can learn both \(A\) and \(\omega\) by sampling \(f\) at just 51 different points \(\in [-\pi, \pi]\).
(a) **THM 2/4:** Let 
\[ \tilde{c}_n := \frac{(-1)^n}{N} \sum_{k=0}^{N-1} f(-\pi + k \cdot \frac{2\pi}{N}) e^{-2\pi i nk / N}. \]
Then,
\[ \tilde{c}_n = \sum_{q=-\infty}^{\infty} c_{n+Nq} = \sum_{m \equiv n \mod N} c_m, \]
where \( c_n \) is the \( n \)th Fourier series coefficient of \( f : [-\pi, \pi] \to \mathbb{C} \).

**HINT:** You will want to use 6 sets of equally spaced samples in \([−\pi, \pi]\), each associated with a different prime number.

5. Do problems 6 and 7 on page 68 of Folland.
6. Do problems 8 and 9 on page 68 of Folland.
7. Do problems 1 and 2 on page 71 of Folland.
8. Do problem 3 on page 71 of Folland.
10. Do problems 6 and 7 on page 71 of Folland.