Do any 7 of the following 10 exercises of your choice. Write up your solutions neatly in your own handwriting! Most problems are from the book, which is available online here: http://www-elec.inaoep.mx/~rogerio/FourierAnalysisUno.pdf.

1. Prove the following facts about the Dirichlet kernel,

\[ D_N(y) := \frac{1}{2\pi} \sin \left( \left( N + \frac{1}{2} \right) y \right) = \frac{1}{2\pi} \sum_{k=-N}^{N} e^{iky}, \]

that we discussed in class. As usual, you should show your work.

(a) Prove that \( D_N(0) = \frac{2N+1}{2\pi} \).

(b) Prove that

\[ D_N(\pi) = \begin{cases} \frac{1}{2\pi} & \text{if } N \text{ is even} \\ -\frac{1}{2\pi} & \text{if } N \text{ is odd} \end{cases}. \]

(c) Prove that the Dirichlet has kernel has exactly 2N zeros in \([−π, π]\). What are they? (Derive an equation for them.)

2. A function \( f : [−π, π] \rightarrow \mathbb{C} \) can always be split into its imaginary and real parts, \( f_1 : [−π, π] \rightarrow \mathbb{R} \) and \( f_2 : [−π, π] \rightarrow \mathbb{R} \), such that

\[ f(x) = f_1(x) + if_2(x) \]

holds for all \( x \in [−π, π] \). Prove the following facts about \( f : [−π, π] \rightarrow \mathbb{C} \).

(a) Prove that the Fourier series coefficients, \( c_n \), of \( f : [−π, π] \rightarrow \mathbb{C} \) always satisfy \( c_n = \bar{c}_{-n} + i\bar{c}_{n} \), where \( \bar{c}_n \) and \( \bar{c}_n \) denote the Fourier series coefficients of \( f_1 \) and \( f_2 \), respectively. This should be easy (i.e., don’t think too hard).

(b) We will say that a function \( f : [−π, π] \rightarrow \mathbb{C} \) is Riemann integrable if both its real and imaginary parts, \( f_1 : [−π, π] \rightarrow \mathbb{R} \) and \( f_2 : [−π, π] \rightarrow \mathbb{R} \), are Riemann integrable. Prove that the real-valued function \( |f(x)|^2 \) is Riemann integrable on \([−π, π]\) whenever \( f : [−π, π] \rightarrow \mathbb{C} \) is Riemann integrable.

(c) Prove that a Riemann integrable real-valued function \( f : [−π, π] \rightarrow \mathbb{R} \) always has Fourier coefficients that satisfy \( c_n = \bar{c}_{-n} \) for all \( n \in \mathbb{Z} \). (Here the bar over \( c_{-n} \) represents complex conjugation.) What can we conclude about \( c_0 \)?

3. Problem 1 on page 42 of Folland.

4. Problem 2 on page 42 of Folland.

5. Problem 5 on page 43 of Folland.

6. Problem 7 on page 43 of Folland.

7. Problem 5 on page 48 of Folland.

8. Problem 8 on page 48 of Folland.
