1. Prove that the following system of equations has no integer solutions.

\[ 11x - 5y = 7 \]
\[ 9x + 10y = -3 \]

*Hint:* Consider each of the equations mod 5.

*Solution:* Assume, by way of contradiction that \((x, y) \in \mathbb{Z}^2\) is a solution to the above system then

\[ x \equiv 2 \mod 5 \]
\[ 4x \equiv 2 \mod 5 \]

The first equation (multiplying both sides by 4) implies \(4x \equiv 3 \mod 5\), which contradicts the second equation. Thus the assumption that the system has an integer solution leads to a contradiction. Therefore, we can conclude that the above system has no integer solutions.

2. Prove: An integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

*Solution:* Let \(x \in \mathbb{Z}\) and \(a_i\) be the \(i^{th}\) digit in its decimal representation, i.e., \(x = \sum_{i=0}^{k} a_i10^i\), where \(0 \leq a_i \leq 9\) for all \(i = 0, \ldots, k\). Note that \(10^i \equiv 1 \mod 3\) for all \(i \in \mathbb{N}\). Thus,

\[ x \equiv \sum_{i=0}^{k} a_i \mod 3. \]

Use this to show that \(3 \mid x\) if and only if \(3 \mid \sum_{i=0}^{k} a_i\).

3. 

(a) Find the multiplicative inverse of each nonzero element in \(\mathbb{Z}_7\). 

*Solution:*

\[ \bar{1}^{-1} = \bar{1}, \quad \bar{2}^{-1} = \bar{4}, \quad \bar{3}^{-1} = \bar{5}, \quad \bar{4}^{-1} = 2, \quad \bar{5}^{-1} = 3, \quad \bar{6}^{-1} = 6. \]

Check that, for example, \(\bar{2} \cdot \bar{4} = \bar{1}\).

(b) Does every nonzero element in \(\mathbb{Z}_{12}\) have a multiplicative inverse?

(c) Formulate a conjecture about which elements in \(\mathbb{Z}_{12}\) will have a multiplicative inverse and which won’t by considering the case of \(\mathbb{Z}_4\) and \(\mathbb{Z}_6\).

(d) Can you generalize your conjecture to \(\mathbb{Z}_n\) for any \(n \in \mathbb{N}, n \geq 2\)?

*Solution:* If \(\gcd(x, n) \neq 1\), then \(x\) will not have a multiplicative inverse in \(\mathbb{Z}_n\).

Have students reach this conjecture on their own. It might help if the construct the multiplication tables for \(\mathbb{Z}_4\) and \(\mathbb{Z}_6\) and see which elements have multiplicative inverses.
(e) Find the multiplicative inverse of 7 mod 31. \((\text{Hint: Use the Euclidean Algorithm to solve } 7x + 31y = 1 \text{ for integers } (x, y))\).

Solution:

\[
\begin{align*}
31 &= 7 \cdot 4 + 3 \\
7 &= 3 \cdot 2 + 1 \\
3 &= 1 \cdot 3 + 0
\end{align*}
\]

Therefore, going backward, we find 
\(1 = 7 - 2 \cdot 3 = 7 - 2(31 - 7 \cdot 4) = 7 \cdot 9 + 31 \cdot (-2)\).
Thus, \(\bar{7} = 9\).

4. For \((a, b), (c, d) \in \mathbb{R}^2\) define \((a, b) \simeq (c, d)\) to mean \(a^2 + b^2 = c^2 + d^2\). Prove that \(\simeq\) is an equivalence relation in \(\mathbb{R}^2\), i.e. prove that it satisfies \text{reflexivity}, \text{symmetry} \text{ and } \text{transitivity}.

Solution:

Reflexivity: We need to show \((a, b) \simeq (a, b) \forall (a, b) \in \mathbb{R}^2\). This is clear, since it is equivalent to \(a^2 + b^2 = a^2 + b^2\).

Symmetry: We need to show that if \((a, b) \simeq (c, d)\) then \((c, d) \simeq (a, b)\). This is immediately evident using the definition of \(\simeq\) and the symmetry of 
\(\text{“} = \text{“}\).

Transitivity: We need to show that if \((a, b) \simeq (c, d)\) and \((c, d) \simeq (e, f)\), then \((a, b) \simeq (e, f)\). Assume \((a, b) \simeq (c, d)\) and \((c, d) \simeq (e, f)\), then, by definition, \(a^2 + b^2 = c^2 + d^2\) and \(c^2 + d^2 = e^2 + f^2\). By transitivity of 
\(\text{“} = \text{“}\) this implies \(a^2 + b^2 = e^2 + f^2\), which in its turn means that \((a, b) \simeq (e, f)\).

5. Let \(S\) be a set with an equivalence relation \(\simeq\), and let \([a]\) denote the class of \(a\) (sometimes denoted as \(\overline{a}\)). Recall the Theorem: We have \([a] = [b]\) if and only if \([a] \cap [b] \neq \emptyset\).

For \((a, b), (c, d) \in \mathbb{R}^2\) define the equivalence relation \((a, b) \simeq (c, d)\) by: \(a^2 + b^2 = c^2 + d^2\). Use the Theorem to prove: \(\text{a. } [(0, 2)] = [(1, \sqrt{3})] \quad \text{b. } [(0, 2)] \cap [(1, 1)] = \emptyset\)

Solution:

\text{a.} Note that \(0^2 + 2^2 = 1^2 + (\sqrt{3})^2\), therefore, \((0, 2) \simeq (1, \sqrt{3})\). By definition of equivalence class, this implies that \((1, \sqrt{3}) \in [(0, 2)]\). Thus, \((1, \sqrt{3}) \in [(0, 2)] \cap [(1, \sqrt{3})]\), in particular, \([(0, 2)] \cap [(1, \sqrt{3})] \neq \emptyset\). The above theorem implies that \([(0, 2)] = [(1, \sqrt{3})]\).

\text{b.} Note that \(0^2 + 2^2 \neq 1^2 + 1^2\), therefore, \((0, 2) \not\sim (1, 1)\). By definition of equivalence class, this implies that \((1, 1) \notin [(0, 2)]\) and therefore \([(0, 2)] \neq [(1, 1)]\).

The theorem contains a biconditional statement, one of the directions asserts that if \([a] \cap [b] \neq \emptyset\), then \([a] = [b]\). The contrapositive of this statement is as follows. “If \([a] \neq [b]\), then \([a] \cap [b] = \emptyset\).”

Since we have shown that \([(0, 2)] \neq [(1, 1)]\), this implies \([(0, 2)] \cap [(1, 1)] = \emptyset\).