Tangent linear function. The geometric meaning of the derivative $f'(a)$ is the slope of the tangent to the curve $y = f(x)$ at the point $(a, f(a))$. The tangent line is itself the graph of a linear function $y = L(x)$, where:

$$L(x) = f(a) + f'(a)(x-a).$$

This is correct because the line $y = f(a) + f'(a)(x-a)$ has slope $m = f'(a)$, and $L(a) = f(a) + f'(a)(a-a) = f(a)$, so the line passes through the point $(a, L(a)) = (a, f(a))$.

The value $f'(a)$ is not just the slope of the tangent line: it is also the slope of the graph itself, because as we zoom in toward $(a, f(a))$, the graph and the tangent line become indistinguishable*:

This suggests a further numerical meaning of the derivative: any function $f(x)$ is very close to being a linear function near a differentiable point $x = a$, so that $L(x)$ is an excellent approximation for $f(x)$ when $x$ is close to $a$:

$$f(x) \approx L(x) = f(a) + f'(a)(x-a) \quad \text{for } x \text{ close to } a.$$  

Much later in §11.10 of Calculus II, we will study Taylor series, which give much better, higher-order approximations to $f(x)$.

EXAMPLE: Find a quick approximation for $\sqrt{1.1}$ without a calculator. Clearly, this is close to $\sqrt{1} = 1$, but we want better. Take $f(x) = \sqrt{x}$, so $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(1) = \frac{1}{2}$. For $x$ near $a = 1$, we have the linear function:

$$L(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1),$$

and the linear approximation:

$$\sqrt{1.1} = f(1.1) \approx L(1.1) = 1 + \frac{1}{2}(0.1) = 1.05.$$  

A calculator gives: $\sqrt{1.1} \approx 1.049$, so our answer is correct to 2 decimal places with very little work. Furthermore, we get approximations for all other square roots near 1 for free, for example $\sqrt{0.96} \approx 1 + \frac{1}{2}(0.96-1) = 1-0.02 = 0.98$.

*By contrast, if we zoom in toward a non-differentiable point, such as $(0,0)$ for the graph $y = |x|$, the graph does not look more and more linear, but rather keeps its angular appearance.
**Example:** Approximate $\sin(42^\circ)$ without a scientific calculator. This is clearly close to $\sin(45^\circ) = \frac{\sqrt{2}}{2} \approx 0.71$, so let us take $a = 45^\circ$. Now, to use calculus with trig functions, we must always convert to radians: $a = 45\left(\frac{\pi}{180}\right) = \frac{\pi}{4}$ rad. Thus $f'(a) = \sin'(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, and we have the linear function:

$$L(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}).$$

The linear approximation is:

$$\sin(42^\circ) = \sin(42\left(\frac{\pi}{180}\right)) \approx L(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} (42\left(\frac{\pi}{180}\right) - \frac{\pi}{4}) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} (\frac{\pi}{60}) \approx 0.67.$$

A scientific calculator gives $\sin(42^\circ) \approx 0.669$, so again the linear approximation is accurate to two decimal places.

**Error sensitivity.** We can rewrite the linear approximation $f(x) \approx f(a) + f'(a)(x-a)$ as:

$$\Delta f = f(x) - f(a) \approx f'(a)(x-a) = f'(a) \Delta x.$$

That is, we can approximate the change in $f(x)$ away from $f(a)$ in proportion to the change in $x$ away from $a$. In Leibnitz notation, with $y = f(x)$, we write this as:

$$\Delta y \approx \frac{dy}{dx} \Delta x.$$

Here we mean $\frac{dy}{dx} = \frac{dy}{dx}|_{x=a} = f'(a)$. If we think of $\Delta x$ as an error from an intended input value $x = a$, then $\Delta f \approx f'(a) \Delta x$ approximates the error from the intended output $f(a)$.

**Example:** A disk of radius $r = 5$ cm is to be cut from a metal sheet weighing $3 \text{ g/cm}^2$. If the radius is measured to within an error of $\Delta r = \pm 0.2$ cm, what is the approximate range of error in the weight? This is the kind of error-control problem from our limit analyses in Notes §1.7, only now we have the powerful tools of calculus to give a simple answer.

The weight is given by the function:

$$W = W(r) = 3\pi r^2$$

and we aim to find the error $\Delta W$ away from this intended value. Since:

$$\frac{dW}{dr} = 3\pi(2r) = 6\pi r$$

and $\frac{dW}{dr}|_{r=5} = 30\pi$, we have the approximate error:

$$\Delta W \approx \frac{dW}{dr}\Delta r = 30\pi \Delta r.$$

Thus, for $\Delta r = \pm 0.2$, we have $\Delta W \approx 30\pi(0.2) \approx 18.8$. That is:

$$r = 5 \pm 0.2 \text{ cm} \implies W \approx 235.6 \pm 18.8 \text{ g}.$$

The point here is not just the specific error estimate, but the formula which gives, for any small input error $\Delta r$, the resulting output error $\Delta W \approx 30\pi \Delta r \approx 94 \Delta r$. The coefficient $30\pi$ measures the sensitivity of the output $W$ to an error in the input $r$.

**Differential notation.** For $y = f(x)$, we rewrite a small $\Delta x$ as $dx$, and we define:

$$dy = \frac{dy}{dx} \ dx \quad \text{and} \quad df = f'(x) \ dx.$$
The dependent variable $dy$ is called a differential: we can think of it as the linear approximation to $\Delta y$, as pictured below:

**Example:** We can rewrite the approximation in the previous example as:

$$\Delta W \approx dW = \frac{dW}{dr} dr = \frac{d}{dr}(3\pi r^2) dr = 6\pi r dr.$$ 

Here $dr$ is just another notation for $\Delta r$, and the approximation $\Delta W \approx 6\pi r \Delta r$ is valid near any particular value of $r$, such as $r = 5$ in the example.

**Linear Approximation Theorem.** How close is the approximation $\Delta y \approx dy$, or equivalently $f(x) \approx L(x) = f(a) + f'(a)(x-a)$? In fact, the difference between $f(x)$ and $L(x)$ is not only small compared to $\Delta x = x-a$, but comparable to $(\Delta x)^2 = (x-a)^2$, which becomes tiny as $\Delta x \to 0$.

Also, the slower the derivative $f'(x)$ changes near $x = a$, the closer $y = f(x)$ is to being linear, and this is measured by the rate of change of $f''(x)$, namely the second derivative $f''(x)$. The following theorem gives an upper bound on the error in the linear approximation, $\varepsilon(x) = f(x) - L(x)$.

**Theorem:** Suppose $f(x)$ is a function such that $|f''(x)| < B$ on the interval $x \in [a-\delta, a+\delta]$. Then, for all $x \in [a-\delta, a+\delta]$, we have:

$$f(x) = f(a) + f'(a)(x-a) + \varepsilon(x), \quad \text{where} \quad |\varepsilon(x)| < \frac{B}{2}|x-a|^2.$$ 

We give the proof, based on the Mean Value Theorem, later in §3.2.

**Example:** For $f(x) = \sqrt{x}$ near $x = 1$, we have $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(1) = \frac{1}{2}$. Also $f''(x) = -\frac{1}{4}x^{-3/2}$, and on the interval $x \in [0.9, 1.1]$, we have:

$$|f''(x)| \leq |f''(0.9)| = \frac{1}{4}(0.9)^{-3/2} \approx 0.29 < 1.$$ 

Thus we may take $B = 1$, and find that:

$$\sqrt{x} = 1 + \frac{1}{2}(x-1) + \varepsilon(x), \quad \text{where} \quad |\varepsilon(x)| < \frac{1}{2}|x-1|^2.$$ 

For example, the error at $x = 1.1$ is $|\varepsilon(1.1)| < \frac{1}{2}(0.1)^2 = 0.005$, so:

$$\sqrt{1.1} = 1 + \frac{1}{2}(0.1) \pm 0.005 = 1.05 \pm 0.005.$$