One of the most basic features of a function is whether it is continuous. Roughly, this means that a small change in \( x \) always leads to a fairly small change in \( f(x) \), without instantaneous jumps. In physical terms, the position of a particle moving in space is continuous, but the position displayed in a video could have a gap, making the position function jump discontinuously. This can be made precise by saying that near \( x = a \), the limit of \( f(x) \) is \( f(a) \):

**Definition:** A function \( f(x) \) is continuous at \( x = a \) whenever \( \lim_{x \to a} f(x) = f(a) \).

Graphically, a function is continuous whenever the graph \( y = f(x) \) proceeds through the point \((a, f(a))\) without jumps or holes.

**Types of discontinuity.** If \( f(x) \) is defined near \( x = a \), continuity can fail in several ways:

i. Removable discontinuity: \( f(a) \) is undefined, but \( \lim_{x \to a} f(x) \) exists.

ii. Removable discontinuity: \( f(a) \) is the “wrong” value, not \( \lim_{x \to a} f(x) \).

iii. Jump discontinuity: the left and right limits are unequal, \( \lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \).

iv. Vertical asymptote: \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to a^-} f(x) \) are \( \pm \infty \).

v. Oscillation discontinuity: \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to a^-} f(x) \) do not exist.

We say \( f(x) \) is continuous on an interval whenever it is continuous at each point of the interval. For the endpoints of a closed interval* \( x \in [a, b] \), we cannot take two-sided limits within the interval, so we only require \( \lim_{x \to a^+} f(x) = f(a) \) and \( \lim_{x \to b^-} f(x) = f(b) \).

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*The interval \([a, b]\) is the set or collection of all numbers \( x \) between \( a \) and \( b \), including the endpoints. The notation \( x \in [a, b] \) means \( x \) is an element of the set \([a, b]\), meaning it is one of the numbers between \( a \) and \( b \), which means \( a \leq x \leq b \).
Proving continuity. As an example, we prove that the absolute value function \( f(x) = |x| \) is continuous at every point \( x = a \). Intuitively, if we move \( x \) by a small amount, then \( |x| \) only changes slightly. This is also clear from the graph \( y = |x| \), which has none of the above discontinuities: the corner at \((0,0)\) is a continuous point, since the graph has no break.

Algebraically, we must check that \( \lim_{x \to a} f(x) = f(a) \) for all \( a \). Now, \( f(x) = x \) for \( x \geq 0 \) and \( f(x) = -x \) for \( x \leq 0 \), so if \( a > 0 \), then \( f(x) = x \) for all \( x \) sufficiently close to \( a \), and \( \lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a) \). Similarly for \( a < 0 \). Finally, for \( a = 0 \), we have the one-sided limits \( \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0 \), and \( \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0 \), which together show the two-sided limit \( \lim_{x \to 0} f(x) = 0 = f(0) \).

Domain of continuity. Almost all functions defined by formulas are continuous, except at points where they are undefined. This follows from our methods for computing limits.

EXAMPLE: Find the points where the following function is continuous:

\[
g(x) = \frac{(x^2-3x+1)\sqrt{x+1}}{x-3}.
\]

First, we consider the factors outside the square root, repeatedly applying the Limit Laws from §1.6:

\[
\lim_{x \to a} \frac{x^2 - 3x + 1}{x - 3} = \frac{(\lim_{x \to a} x)^2 - 3 (\lim_{x \to a} x) + 1}{(\lim_{x \to a} x) - 3} = \frac{a^2 - 3a + 1}{a - 3},
\]

provided the denominator \( a - 3 \) is non-zero; that is, \( a \neq 3 \). The Limit Laws also give \( \lim_{x \to a} \sqrt{x+1} = \sqrt{a+1} \) provided \( a+1 > 0 \) to avoid the square root of a negative number; that is, for \( a > -1 \). Combining these, we have:

\[
\lim_{x \to a} \frac{(x^2-3x+1)\sqrt{x+1}}{x-3} = \lim_{x \to a} \frac{x^2 - 3x + 1}{x - 3} \cdot \lim_{x \to a} \sqrt{x+1} = \frac{(a^2-3a+1)\sqrt{a+1}}{a - 3},
\]

provided both factor limits exist, that is if \( a \neq 3 \) and \( a > -1 \). That is, \( g(x) \) is continuous for all these values of \( a \). The remaining values are:

- \( a < -1 \), where \( g(x) \) is undefined, hence not continuous;
- \( a = -1 \), where \( g(x) \) is continuous, since at the left endpoint of the domain of definition, we only require the one-sided limit \( \lim_{x \to a^+} g(x) = g(a) \);
- \( a = 3 \), where the function clearly has a vertical asymptote, discontinuity of type (iv).

In summary, our \( g(x) \) is continuous at every point where it is defined, that is, in the intervals\(^\dagger\) \([-1,3) \cup (3,\infty)\). The graph looks like:

\(^\dagger\)The half-open interval \([a,b)\) is the set of all numbers \( x \) between \( a \) and \( b \), including the left endpoint \( x = a \) but excluding the right endpoint \( x = b \); that is, \( a \leq x < b \). The infinite interval \((a,\infty)\) means all \( x > a \), with \( \infty \) indicating no upper bound on the right.
Intermediate Value Theorem:

Composing continuous functions. Another way to combine functions \( f(x) \) and \( g(x) \) is to compose or chain them, taking the output of \( g \) as the input of \( f \) to obtain the new function \( f(g(x)) \). Composition also preserves continuity: if \( g(x) \) is continuous at \( x = a \), and \( f(x) \) is continuous at \( x = g(a) \), then \( f(g(x)) \) is continuous at \( x = a \). This follows from the following theorem:

Composition Law: We have:

\[
\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)),
\]

provided \( \lim_{x \to a} g(x) = b \) and \( f(x) \) is continuous at \( x = b \).

Proof. For any desired output error bound \( \varepsilon > 0 \), we must find some input accuracy \( \delta > 0 \) such that \( |x - a| < \delta \) forces \( |f(g(x)) - f(b)| < \varepsilon \).

Take any \( \varepsilon > 0 \). Since \( f(y) \) is continuous at \( y = b \), there is \( \delta_1 > 0 \) such that \( |y - b| < \delta_1 \) forces \( |f(y) - f(b)| < \varepsilon \). Also, since \( \lim_{x \to a} g(x) = b \), there exists \( \delta > 0 \) such that \( 0 < |x - a| < \delta \) forces \( |g(x) - b| < \delta_1 \). Therefore \( 0 < |x - a| < \delta \) forces \( |g(x) - b| < \delta_1 \), which in turn forces \( |f(g(x)) - f(b)| < \varepsilon \), as required.

Intermediate Value Theorem:

If \( f(x) \) is continuous for \( x \) in the interval \([a,b]\), and \( r \) is between \( f(a) \) and \( f(b) \), meaning either \( f(a) < r < f(b) \) or \( f(a) > r > f(b) \), then there is a value\(^4\) \( c \in (a,b) \) such that \( f(c) = r \).

This says that as the function value \( f(x) \) goes continuously from \( f(a) \) to \( f(b) \), perhaps rising and falling many times, it must pass through every value \( r \) between \( f(a) \) and \( f(b) \).

Note that this is not necessarily true for a discontinuous function like \( g(x) \) in the graph above: taking \([a,b] = [2,4]\), we have \( g(2) \approx 1.7 \), \( g(4) \approx 11.2 \), and \( g(2) < 7 < g(4) \), but there is a vertical asymptote discontinuity in the interval \([2,4]\), and there is no \( c \in (2,4) \) with \( g(c) = 7 \).

However, \( g(x) \) is continuous over the interval \([0,1]\), with \( g(0) \approx -0.33 \), \( g(1) \approx 0.72 \), and \( g(0) < 0 < g(1) \), so the Theorem says there must be some \( c \in (0,1) \) with \( g(c) = 0 \). This is just the \( x \)-intercept visible in the graph.

Example: Show that there exists a solution \( x = c \) to the equation \( \cos(x) = x \). We have no easy way of solving this equation, but writing \( f(x) = \cos(x) - x \), we know that \( f(0) = 1 \), \( f(\pi) = -1 - \pi \), and \( f(0) > 0 > f(\pi) \). Since \( f(x) \) is continuous, the Theorem guarantees some \( c \in [0,\pi] \) with \( f(c) = 0 \), meaning \( \cos(c) = c \).

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\(^4\)Here \( c \) lies in the open interval \( (a,b) \), between \( a \) and \( b \) but excluding both endpoints: \( a < c < b \).