Section 4.5

Exercise 4.5.3: There is no continuous function \( f : \mathbb{R} \to \mathbb{R} \) with \( f(\mathbb{R}) = \mathbb{Q} \). If there were, then we could find \( a, b \in \mathbb{R} \) with \( f(a) = 1 \) and \( f(b) = 2 \). Either \( a < b \) or \( b < a \). Let’s suppose \( a < b \). Since \( f(x) \) is continuous on \( \mathbb{R} \) it is also continuous on \([a, b] \). By the intermediate value theorem, and the fact that \( 1 < \sqrt{2} < 2 \), there exists a \( c \in (a, b) \) such that \( f(c) = \sqrt{2} \). But \( \sqrt{2} \notin \mathbb{Q} \), hence \( f(\mathbb{R}) \), which includes \( f(c) \), cannot just be \( \mathbb{Q} \). The same argument works if \( b < a \), but we find \( c \in (b, a) \) instead.

Exercise 4.5.4: Let \( f(x) : [a, b] \to \mathbb{R} \) be an increasing function \( (x < y \implies f(x) \leq f(y)) \) such that for all \( x < y \) in \([a, b] \) and \( L \in \mathbb{R} \) between \( f(x) \) and \( f(y) \) there is a \( cL \in (x, y) \) with \( f(cL) = L \). Pick \( v \in [a, b] \). We wish to show that \( f(x) \) is continuous at \( v \). Pick \( \epsilon > 0 \). Now let \( x = a \) and \( y = v \). Since \( f(x) \) is increasing on \([a, v] \) we have \( h = f(v) - f(a) \geq 0 \). Use the intermediate value property to find \( d_1 \) so that \( f(d_1) = f(v) - \hat{h} \) where \( \hat{h} = \frac{1}{2} \min\{h, \epsilon\} \). (Note: if \( h = 0 \), then \( f(x) = f(v) \) for every \( x \in (a, v) \), so \( d_1 \) can be any value in \((a, v)\). Then for all \( x \in (d_1, v) \) we have \( f(d_1) \leq f(x) \leq f(v) \), i.e. \( f(v) - \hat{h} \leq f(x) \leq f(v) \). However, \( \hat{h} < \epsilon \) so this implies \(-\epsilon < f(x) - f(v) \leq 0 \) for all \( x \in (d_1, v) \). We now do the same on the other side. Let \( \hat{h} = \frac{1}{2} \min\{f(b) - f(v), \epsilon\} \). Then find \( d_2 \) such that \( f(d_2) = f(v) + \hat{h} \). Again we’ll find that for \( x \in (v, d_2) \) we have \( f(v) \leq f(x) \leq f(v) + \hat{h} < f(v) + \epsilon \). Putting the two sides together, we get for \( x \in (d_1, d_2) \) \(-\epsilon < f(x) - f(v) \leq \epsilon \) which is just \( |f(x) - f(v)| \leq \epsilon \). Now just choose \((v - \delta, v + \delta) \subset (d_1, d_2) \), and we have shown that \( f(x) \) is continuous at \( v \).

Exercise 4.5.7: First, note that \( f([0, 1]) \subset [0, 1] \) implies \( f(0) \geq 0 \) and \( f(1) \leq 1 \). If either \( f(0) = 0 \) or \( f(1) = 1 \), we have found a fixed point. The remaining case to consider is when \( f(0) > 0 \) and \( f(1) < 1 \). Let \( g(x) = f(x) - x \). Then \( g(x) \) is continuous on \([0, 1] \) since \( f(x) \) and \( y = x \) are continuous on \([0, 1] \). In the case we are considering, \( g(0) = f(0) - 0 > 0 \) and \( g(1) = f(1) - 1 < 0 \). By the intermediate value theorem, there is a point \( c \in (0, 1) \) with \( g(c) = 0 \), since \( g(1) < 0 < g(0) \). But for this value of \( c \) \( g(c) = 0 \) means \( f(c) - c = 0 \) or \( f(c) = c \).

Section 5.2

Exercise 5.2.2:

(i) Let \( f(x) = \frac{1}{x} \). To compute the derivative, \( f'(a) \), for \( a \neq 0 \) we evaluate

\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{1/x - 1/a}{x - a} = \lim_{x \to a} \frac{a-x}{ax} = \lim_{x \to a} \frac{a - x}{ax(a-x)} = \lim_{x \to a} \frac{1}{xa} = \frac{1}{a^2}
\]

which is in agreement with the usual formula: \( \frac{d}{dx} x^n = nx^{n-1} \) for \( n = -1 \). Thus \( f(x) \) is differentiable on \((-\infty, 0) \cup (0, \infty) \).

(ii) First, use the product rule for \( f(x) \cdot h(x) \) where \( h(x) = \frac{1}{g(x)} \). Note that the domain of \( h(x) \) is exactly the subset of the domain of \( g(x) \) where \( g(x) \neq 0 \). Provided \( g(c) \neq 0 \) we can consider the value of \( f(x)h(x) \) at \( c \). The product rule tells us that if \( f(x) \) and \( h(x) \) are both differentiable at \( x = c \) then so is \( f(x)h(x) \) with derivative \( f'(c)h(c) + f(c)h'(c) \). However, we do not yet know that \( h(x) \) is differentiable at \( x = c \). This is where we use the chain rule. \( g(x) \) is differentiable at \( x = c \) and since \( g(c) \neq 0 \), \( g(x) = \frac{1}{x} \) is differentiable at \( g(c) \). The chain rule then tells us that \( g(g(x)) = \frac{1}{g(x)} \) is differentiable at \( x = c \) with derivative \( g'(g(c))g'(c) \).

By the preceding part \( g'(c) = \frac{-1}{g(c)^2} \). So \( h'(c) = \frac{-g'(c)}{g(c)^2} \). Putting these two together gives

\[
\frac{d}{dx} \frac{f(x)}{g(x)} \bigg|_{x=c} = f'(c)g'(c) - f(c)g'(c) = f'(c)g(c) - f(c)g'(c) \bigg/ g(c)^2
\]

(iii) To directly verify the formula use

\[
\lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} = \lim_{x \to c} \frac{f(x)(g(x)-g(c)) + g(c)(f(x)-f(c))}{x-c} = \frac{f(c)g'(c) - f'(c)g(c)}{g(c)(x-c)}
\]
Rewrite \( f(x)g(c) - g(x)f(c) = (f(x) - f(c))g(c) - (g(x) - g(c))f(c) \), so that
\[
\frac{d}{dx} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{1}{g(x)g(c)} \left( \frac{f(x) - f(c)}{x - c} g(c) - \frac{g(x) - g(c)}{x - c} f(c) \right)
\]
From here it is just a matter of calculating the limit, noting that \( \lim_{x \to c} g(x) = g(c) \) since, \( g(x) \) being differentiable at \( x = c \), \( g(x) \) is also continuous at \( x = c \). The rest of the limits follow from the definition of \( f'(c) \) and \( g'(c) \) and the algebraic limit theorem.

**Exercise 5.2.3:** Consider the following function
\[
f(x) = \begin{cases} 
x^2 & x \in \mathbb{Q} \\
0 & x \notin \mathbb{Q}
\end{cases}
\]
We first verify that the function is differentiable at \( x = 0 \) where \( f(x) = 0 \).
\[
\lim_{x \to 0} \frac{f(x) - 0}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}
\]
But \( 0 \leq \left| \frac{f(x)}{x} \right| \leq \left| \frac{x^2}{x} \right| = |x| \), so as \( x \to 0 \) we must have
\[
\lim_{x \to 0} \frac{f(x)}{x} = 0
\]
Thus \( f'(0) = 0 \). To see that the function is not differentiable anywhere else we will show that it is not even continuous except when \( x = 0 \). To that end let \( c \neq 0 \). If \( c \in \mathbb{Q} \), let \( x_n \to c \) be a sequence of irrationals converging to \( c \). Then \( \lim_{n \to \infty} f(x_n) = 0 \) but \( f(c) = c^2 \neq 0 \). If \( c \notin \mathbb{Q} \), use a sequence of rationals \( q_n \to c \). Then \( \lim_{n \to \infty} f(q_n) = \lim_{n \to \infty} q_n^2 = c^2 \) (since \( y = x^2 \) is continuous at \( x = c \)), but \( f(c) = 0 \).

**Exercise 5.2.5:** First note that
\[
g'_a(x) = a x^{a-1} \sin \left( \frac{1}{x} \right) - x^{a-2} \cos \left( \frac{1}{x} \right)
\]
when \( x \neq 0 \). For \( x = 0 \) we need to evaluate
\[
\lim_{x \to 0} \frac{g(x) - 0}{x - 0} = \lim_{x \to 0} x^{a-1} \sin \left( \frac{1}{x} \right)
\]
This limit will exist and equal 0 provided \( a > 1 \) and \( x^{a-1} \) is defined on \( \mathbb{R} \) (no square roots allowed, for example).

(i) Note that \( g(x) \) differentiable at \( x = 0 \) means that the first part of \( g(x) \) will have a limit as \( x \to 0 \), and otherwise \( g'(x) \) is bounded. So to get something unbounded requires that we make the second part unbounded, and that means \( x^{a-2} \) must be unbounded. This happens when \( a < 2 \). So we need \( 1 < a < 2 \) to meet both conditions. Choose \( a = \frac{5}{3} \), say (we use a cubed root to ensure that \( g_a(x) \) is defined on all of \( \mathbb{R} \).

(ii) To have \( g'_a(x) \) continuous we need to have \( a > 2 \) so that both terms will go to 0 as \( x \to 0 \), and the derivative at 0 will also equal 0. This will make \( x^{a-1} \) have exponent bigger than 1 which will allow the first term of \( g'_a(x) \) to have a derivative at 0 equal to 0, but as long as \( a - 2 \leq 1 \) the second term will not have a derivative at \( x = 0 \). So we choose \( a = \frac{7}{3} \).

(iii) By the reasoning in part (ii), if \( a > 3 \) then \( g'_a(x) \) will have a derivative at \( x = 0 \) and will be differentiable at every other \( x \) as well. Now lets consider the derivative of \( g'_a(x) \). We have for \( x \neq 0 \) that
\[
g''_a(x) = a(a - 1)x^{a-2} \sin \left( \frac{1}{x} \right) - a x^{a-3} \cos \left( \frac{1}{x} \right) - (a - 2)x^{a-3} \cos \left( \frac{1}{x} \right) + x^{a-4} \sin \left( \frac{1}{x} \right)
\]
Let \( a = 4 \). Then \( g_a(x) \) is differentiable at \( x = 0 \) by our preceding calculations, but \( g''(a) \) will not have a limit as \( x \to 0 \) due to the appearance of \( \sin \left( \frac{1}{x} \right) \).