Section 5.3

Exercise 5.3.1: Let \( f : [a, b] \to \mathbb{R} \) be differentiable with \( f'(x) \) continuous on \([a, b]\). Since \( f'(x) \) is continuous on \([a, b]\) we know that \(|f'(x)|\) is continuous as well, since it is the composition of continuous functions. Since \([a, b]\) is compact and \( f'(x) \) is continuous, there is a point \( c \in [a, b] \) where \(|f'(c)| = M\) is a maximum. Now, the conditions on \( f(x) \) allow an application of the mean value theorem to \( f(x) \) on any interval \([d, e] \subset [a, b]\). This tells us there is another point \( c' \in [d, e] \) with

\[
\frac{f(e) - f(d)}{e - d} = f'(c')
\]

In absolute value, \(|f'(c')| \leq M\) since \( M \) is the maximum. But this implies that for any \( d < e \) in \([a, b]\) we have

\[
\left| \frac{f(e) - f(d)}{e - d} \right| \leq M
\]

Changing the order of \( e \) and \( d \) will not change the term on the left, so this implies that for any \( x, y \in [a, b] \) we will have the desired conclusion (let \( d = x \) and \( e = y \)). Hence \( f(x) \) is Lipschitz.

Exercise 5.3.3: (a) Since \( h(x) \) is differentiable on \([0, 3]\) we know that it is also continuous. Thus \( f(x) = h(x) - x \) is continuous as well on that interval. But \( f(0) = -1 \), \( f(1) = -1 \) and \( f(3) = 1 \) using the values in the problem. By the intermediate value theorem there is a point \( d \in [1, 3] \) where \( f(d) = 0 \) and thus \( h(d) = d \).

(b) The conditions on \( h(x) \) allow us to apply the mean value theorem on the interval \([0, 3]\) since the function is continuous there and differentiable on \((0, 3)\) (in fact on a larger set). But \( h(3) - h(0) = 2 - 1 = 1 \) and \( 3 - 0 = 3 \).

(c) First apply the mean value theorem on \([1, 3]\) to find a point \( c \in (1, 3) \) with \( h'(c) = \frac{1}{3} \). We know that \( e \neq c \), where \( c \) is the point found in part \( b \), since the derivative takes different values at these points. Then Darboux’ theorem tells us that there is a point between \( e \) and \( c \) where \( h' \) must equal \( \frac{1}{4} \) since this value is between \( h'(e) \) and \( h'(c) \), and derivatives have the intermediate value property on closed intervals where they are defined.

Exercise 5.3.5: Suppose that \( f(x) \) has two fixed points, \( d \) and \( e \) with \( d < e \), in an interval where \( f'(x) \) is defined and \( f'(x) \neq 1 \). Then \( f(d) = d \) and \( f(e) = e \). In addition, \( f(x) \) is continuous on \([d, e]\) since it is differentiable on \([d, e]\). Furthermore, \( f(x) \) is differentiable on \((d, e)\) as well. The mean value theorem applies on \([d, e]\) and guarantees the existence of a point \( c \in (d, e) \) with

\[
f'(c) = \frac{f(e) - f(d)}{e - d} = \frac{e - d}{e - d} = 1
\]

which contradicts that \( f'(x) \neq 1 \) on the interval containing \([d, e]\).

Exercise 5.3.7: (a) Let \( f : (a, b) \to \mathbb{R} \) be an increasing function that is also differentiable on \((a, b)\). If \( c \in (a, b) \) then

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}
\]

Let \( x_n \to c \) with \( x_n > c \), then we have that \( f(x_n) - f(c) \geq 0 \) since the function is increasing and \( x_n - c > 0 \). Since the limit above exists, we must have \( f'(c) = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \geq 0 \) since the fraction in this limit is \( \geq 0 \). Thus, for each point \( c \in (a, b) \) we must have \( f'(c) \geq 0 \). Now assume that \( f'(c) \geq 0 \) for each \( c \in (a, b) \). To show that the function is increasing, we need to know that \( x < y \) with \( x, y \in (a, b) \) implies \( f(x) \leq f(y) \). But \( f \) on \([x, y]\) is differentiable and thus continuous, so we may apply the mean value theorem to conclude that there is a \( c \in (x, y) \) with

\[
f'(c) = \frac{f(y) - f(x)}{y - x}
\]

Since \( c \in (x, y) \subset (a, b) \) we have, by assumption, that \( f'(c) \geq 0 \). Furthermore, \( y > x \) makes the denominator positive. Thus the numerator must be \( \geq 0 \), or \( f(y) \geq f(x) \).
(b) For \( g(x) \) in the statement of the problem, we calculate
\[
g'(0) = \lim_{x \to 0} \frac{\frac{x}{2} + x^2 \sin \left( \frac{1}{x} \right) - 0}{x - 0} = \lim_{x \to 0} \frac{1}{2} + x \sin \left( \frac{1}{x} \right) = \frac{1}{2}
\]
Thus \( g'(0) > 0 \). However, the function is not increasing on any interval \((-\delta, \delta)\) containing 0. We see this by computing the derivative of \( g(x) \) on \((-\infty, 0) \cup (0, \infty)\) to get
\[
g'(x) = \frac{1}{2} + 2x \sin \left( \frac{1}{x} \right) - \cos \left( \frac{1}{x} \right)
\]
If this is negative at any point in \((-\delta, \delta)\) then the function is not increasing, as shown in the previous part of the problem. So we choose points where \( \cos \left( \frac{1}{x} \right) \) is as big as possible, namely equal to 1. This occurs when \( \frac{1}{x} = 2k\pi \) for \( k \in \mathbb{Z} \), i.e. for \( x_k = \frac{1}{2k\pi} \). If we let \( k \in \mathbb{N} \), then we obtain a sequence \( x_k \to 0 \) which must then enter \((-\delta, \delta)\).
On the other hand, \( \sin \left( \frac{1}{x_k} \right) = \sin(2k\pi) = 0 \) for all \( k \in \mathbb{N} \), so the middle term will be 0 for each of these points. Thus \( g'(x_k) = \frac{1}{2} + 0 - 1 = -\frac{1}{2} < 0 \). For \( k \) large enough this point will be in \((-\delta, \delta)\).

**Exercise 5.3.8:** Let \( g : (a, b) \to \mathbb{R} \) be differentiable at a point \( c \in (a, b) \). We assume that \( g'(c) > 0 \) (the case where \( g'(c) < 0 \) will be done below). We **cannot** use the mean value theorem since we only know that the function is differentiable at a single point. Instead we use the definition of the derivative. Since
\[
0 < g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}
\]
there is an \( \epsilon > 0 \) such that \( g'(c) - \epsilon > 0 \) and a \( \delta > 0 \) such that \( 0 < |x - c| < \delta \) implies
\[
\left| \frac{g(x) - g(c)}{x - c} - g'(c) \right| < \epsilon
\]
This further implies that
\[
\frac{g(x) - g(c)}{x - c} > g'(c) - \epsilon > 0
\]
When \( 0 < x - c < \delta \) we have that \( g(x) > g(c) \) in order for the fraction to be bigger than 0. When \( 0 > x - c > -\delta \) we have that \( g(x) < g(c) \) so that both numerator and denominator will be negative. In either case, there is a \( \delta \)-neighborhood of \( x = c \) in which \( g(x) \neq g(c) \) for \( x \neq c \). To address the case where \( g'(c) < 0 \), let \( h(x) = -g(x) \). Then \( h'(c) = -g'(c) > 0 \). Thus there is a neighborhood of \( c \) where \(-g(x) \neq -g(c) \) unless \( x = c \). Multiplying by -1 yields \( g(x) \neq g(c) \) on the same neighborhood.

To relate to the previous problem: that \( g'(0) = \frac{1}{2} \) in 5.3.7 tells us that there is no other point in a sufficiently small neighborhood \((-\delta, \delta)\) with \( g(x) = g(0) = 0 \). In fact, all points with \( x \in (0, \delta) \) will have \( g(x) > 0 \) and all points \( x \in (-\delta, 0) \) will have \( g(x) < 0 \). However, there is still room for the function to increase and decrease within these conditions, as seen from 5.3.7 (b).