9. Suppose \( \{v_1, v_2\} \) is a basis for a vector space \( V \). Let \( \{w_1, w_2, w_3\} \) be any set of vectors in \( V \). Prove that \( \{w_1, w_2, w_3\} \) is not a basis for \( V \).

**Answer:**

We will show that \( \{w_1, w_2, w_3\} \) is linearly dependent. As \( \{v_1, v_2\} \) is a basis, we can write:

\[
\begin{align*}
w_1 &= a_1 v_1 + a_2 v_2 \\
w_2 &= b_1 v_1 + b_2 v_2 \\
w_3 &= c_1 v_1 + c_2 v_2
\end{align*}
\]

for \( a_i, b_i, c_i \in \mathbb{F} \).

If \( a_1 = b_1 = c_1 = 0 \), then each of \( \{w_1, w_2, w_3\} \) is a scalar multiple of \( v_2 \), and \( \{w_1, w_2, w_3\} \) is linearly dependent. So we may assume (possibly after reindexing \( \{w_1, w_2, w_3\} \)) that \( a_1 \neq 0 \).

Then

\[
\begin{align*}
w_2 - \frac{b_1}{a_1} w_1 &= \left( b_2 - \frac{b_1}{a_1} a_2 \right) v_2 \\
w_3 - \frac{c_1}{a_1} w_1 &= \left( c_2 - \frac{c_1}{a_1} a_2 \right) v_2
\end{align*}
\]

If \( b = b_2 - \frac{b_1}{a_1} a_2 \) is 0, then \( w_1 \) and \( w_2 \) are linearly dependent.

If \( c = c_2 - \frac{c_1}{a_1} a_2 \) is 0, then \( w_1 \) and \( w_3 \) are linearly dependent.

Therefore we may assume that \( b \neq 0 \neq c \). But then the equation

\[
v_2 = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3
\]

has at least two distinct solutions:

\[
(\alpha_1, \alpha_2, \alpha_3) = \left( \frac{b_1}{a_1} b^{-1}, b^{-1}, 0 \right)
\]

\[
(\alpha_1, \alpha_2, \alpha_3) = \left( -\frac{c_1}{a_1} c^{-1}, 0, c^{-1} \right)
\]

and \( \{w_1, w_2, w_3\} \) is linearly dependent.

Therefore in all cases \( \{w_1, w_2, w_3\} \) is linearly dependent. In particular \( \{w_1, w_2, w_3\} \) is not a basis.