1. You roll two dice. What is the probability of two sixes? Of exactly one 6? Of no sixes?
   
   Answer: 1/36, 10/36 = 5/18, 25/36

2. You roll two dice. What is the expected number of sixes that will show?
   
   Answer: 12/36 = 1/3

3. When all factors are taken into account, an insurance company estimates that the probability of the owners’ of certain house making a claim for $5000 is 0.1, and that the probability of the house and contents being totally destroyed is .005. Should that tragedy happen, the company will have to pay $150,000. The company charges $1300 for the insurance policy. What is the expected value of this policy to the insurance company?
   
   Answer: $50

4. The night watchman in a factory cannot guard both the safe in back and the cash register in front. The safe contains $6000, while the register has only $1000. Tonight the guard fears a robbery; the probability that the thief will try the cash register is 0.8 and the probability the thief will try the safe is 0.2. If the guard is not present, the thief will take all the money. If the guard is present, the thief will go away empty handed. Where should the guard be positioned in order to minimize the thief’s gains?
   
   Answer: at the safe. The expected gain to the thief is $800

5. What is the expected number of rolls of three dice before triples is first obtained?
   
   ans: 36

6. You are taking a 30 question, multiple choice test (five choices per question). The directions say that your grade will equal the number of correct answers minus one-fourth of the number of wrong answers. You are sure you have 20 of the answers correct. On each of the remaining 10 questions, you can definitely eliminate two of the choices. If you choose from the remaining three responses at random, what is your expected grade for the entire test?
   
   Answer: 21 \frac{2}{3}

7. You and your friend are playing the following game: two dice are rolled; if the total showing is divisible by 3, you pay your friend $6. How much should he pay you when the total is not divisible by 3 if you want to make the game fair? A fair game is one in which your expected winnings are $0.
Answer: The probability that the sum is divisible by 3 is $\frac{1}{2}$ and the probability that the sum is not divisible by 3 is $\frac{2}{3}$. You can see this by simply writing down all the possibilities. If $x$ is the amount you should be paid, then

$$\frac{1}{3} \times (-6) + \frac{2}{3} \times x = 0$$

Solve for $x$ and get $x = 3$.

8. A friend offers to play you a game where you pay him $1 if the roll of a 6-sided die comes up as 1, 2, 3, or 4, and he pays you $2 if the die comes up a 5 or 6. What is the expected value of a round (for you) if you play the game? What are the odds of your winning?

Answer: The probability that you pay $1 is $\frac{4}{6}$ since he has 4 ways to win, and the probability that he pays you $2 is $\frac{2}{6}$. In symbols, this is

$P(-1) = \frac{4}{6}$  
$P(2) = \frac{2}{6}$  
so that  
$E = -1 \times (\frac{4}{6}) + 2 \times (\frac{2}{6}) = 0$.

The odds in this case are 2:1 in your friend’s favor. That the expected value is 0 precisely when the payoffs are equal to the odds is a general rule. That is, if the odds of winning a game are $a:b$, then a fair game, means that the first player’s payoff for winning should be $b$ and the second player’s payoff should be $a$ to make the game fair.

9. On flying to and from Denver from Lansing airport, I have calculated the following probabilities (roughly). The probability that I get to Denver on time is $\frac{2}{3}$, the probability that I get to Denver 1 hour late is $\frac{1}{6}$, the probability that I get to Denver 2 hours late is $\frac{1}{12}$, and the probability that I get to Denver 3 hours late is $\frac{1}{12}$. What is my expected value of lateness?

Answer: The problem tells us that we have the following probabilities:

$P(0) = \frac{2}{3}$  
$P(1) = \frac{1}{6}$  
$P(2) = \frac{1}{12}$, and  
$P(3) = \frac{1}{12}$.

Thus the expected value is:

$$E = (0 \times \frac{2}{3}) + (1 \times \frac{1}{6}) + (2 \times \frac{1}{12}) + (3 \times \frac{1}{12}) = \frac{7}{12}.$$  

Thus my expected value of lateness is $\frac{7}{12}$ of an hour, or about 35 minutes.

10. My friend and I play the following game: I pay him $2 to play. I flip a coin at most 5 times. If it comes up heads the first time, I get a $1 and the game ends, otherwise, I flip again and if it comes up heads this time, I get $2. We continue in this way, so that if the first heads is on the third toss I get $4, the fourth toss, $8, and the fifth toss $16. What is my expected value for this game?
Answer: We have the following table for probabilities of when the first heads appears.

\[
\begin{align*}
P(0) &= 1/32 \quad \text{The probability that no heads appear} \\
P(1) &= 1/2 \\
P(2) &= 1/4 \\
P(3) &= 1/8 \\
P(4) &= 1/16 \\
P(5) &= 1/32.
\end{align*}
\]

Thus the expected value of what he pays me back is

\[
0 \times P(0) + 1 \times P(1) + 2 \times P(2) + 4 \times P(3) + 8 \times P(4) + 16 \times P(5) = 0 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 + 1/2 = 2.5.
\]

Thus my expected value is $2.50, or 50 cents.

11. In the game of birdcage, you pay $1 to play the game. If one of the three dice comes up 6, you get $2 back (earning $1). If two dice come up 6, you get $3 back, and if all three dice come up 6, you get $4 back. What is the expected value of the game?

Answer: The probability of no sixes coming up is \(\frac{125}{216}\), and in this case you lose $1. We next need the probability of exactly one 6 coming up. This can happen in one of three ways: the six can be on the first die, the second die, or the third die. For exactly one six, with it on the first die, there are \(1 \times 5 \times 5 = 25\) ways for this to happen. Similarly, there are 25 ways for the six to appear on the second die, and 25 ways for it to appear on the third die (and for exactly one 6). Thus there are 75 ways for exactly one six to appear. The value of this outcome is $1. For exactly two sixes to appear, there are 5 ways for the first two dice to be sixes, 5 ways for the first and the third dice to be sixes, and 5 ways for the last two dice to be sixes. Consequently there are 15 ways for two dice to come up six. The value of this outcome is $2. Finally, there is exactly one way for three sixes to come up, with a value of $3. Thus the expected value (for you) is

\[
\frac{125}{216}(-1) + \frac{75}{216}(1) + \frac{15}{216}(2) + \frac{1}{216}(3) = \frac{-17}{216}.
\]

Hence, for every game you play, you expect to lose 7.8 cents.

12. The game “Who Wants to be a Millionaire” has $100, $200, $500, $1000, $2000, $4000, $8000, $16000, $32000, $64000, $125000, $250000, $500000, and $1000000 questions. You start at the $100 levels and work your way up through the questions if you continue to get right answers. Suppose you got to keep the amount of money of your last correct question (which isn’t what happens in the show). Each question is multiple choice with 4 possible answers, only one of which is correct. What is the expected amount of money that a contestant would get if they guess randomly on each question?

Answer: We have the following probabilities for money amounts
Thus the expected value is:

\[100 \times \left(\frac{1}{4}\right) \times \left(\frac{3}{4}\right) + 200 \times \left(\frac{1}{4}\right) \times \left(\frac{3}{4}\right)^2 + \cdots + 500000 \times \left(\frac{3}{4}\right) \times \left(\frac{1}{4}\right)^{13} + 1000000 \times \left(\frac{1}{4}\right)^{14} = 39.84081.\]

So about $39.84.

13. The prince has searched all over his kingdom for the maiden who left her glass slipper at the ball. He finally arrives at the estate where Cinderella and her two evil step-sisters live. The slipper only fits Cinderella. The prince tries the slipper on the three maidens in random order. What is the expected number of maidens on whom the prince must try the slipper until he finally finds Cinderella?

Solution:

Let \( E_1 \) be the event that the first maiden is Cinderella.

Let \( E_2 \) be the event that the second maiden is Cinderella.

Let \( E_3 \) be the event that the third maiden is Cinderella.

\[ p(E_1) = \frac{1}{3} \quad \text{and} \quad v(E_1) = 1 \]

\[ p(E_2) = \frac{1}{3} \quad \text{and} \quad v(E_2) = 2 \]

\[ p(E_3) = \frac{1}{3} \quad \text{and} \quad v(E_3) = 3 \]

Substitute these values into

\[ E.V. = p(E_1) \times v(E_1) + p(E_2) \times v(E_2) + p(E_3) \times v(E_3) = \frac{1 + 2 + 3}{3} = 2 \]

14. You start with a full deck of cards, which have been shuffled. You draw cards from the deck, without replacement, until you get a card other than an ace. What is the expected value of the number of cards drawn?

Solution:
Let $E_1$ be the event that the first card is not an ace.
Let $E_2$ be the event that the first card is an ace and the second card is not an ace.
Let $E_3$ be the event that the first two cards are aces and the third card is not an ace.
Let $E_4$ be the event that the first three cards are aces and the fourth card is not an ace.
Let $E_5$ be the event that the first cards are aces and hence the fifth card is not an ace.

\[
p(E_1) = \frac{48}{52} \quad \text{and} \quad v(E_1) = 1
\]

\[
p(E_2) = \frac{4 \times 48}{52 \times 51} \quad \text{and} \quad v(E_2) = 2
\]

\[
p(E_3) = \frac{4 \times 3 \times 48}{52 \times 51 \times 50} \quad \text{and} \quad v(E_3) = 3
\]

\[
p(E_4) = \frac{4 \times 3 \times 2 \times 48}{52 \times 51 \times 50 \times 49} \quad \text{and} \quad v(E_4) = 4
\]

\[
p(E_5) = \frac{4 \times 3 \times 2 \times 1 \times 48}{52 \times 51 \times 50 \times 49 \times 48} \quad \text{and} \quad v(E_5) = 1
\]

Substitute these values into

\[
E.V. = p(E_1) \times v(E_1) + p(E_2) \times v(E_2) + p(E_3) \times v(E_3) + p(E_4) \times v(E_4) + p(E_5) \times v(E_5) \approx 1.08163
\]

15. What is the expected number of face cards (jack, queen, or king) in a three card hand drawn at random from a standard deck of cards?

Solution:

Let $E_1$ be the event that there are no face cards
Let $E_2$ be the event that there is exactly one face card.
Let $E_3$ be the event that there are exactly two face cards.
Let $E_4$ be the event that all three cards are face cards.

\[
p(E_1) = \frac{\binom{40}{3}}{\binom{52}{3}} \quad \text{and} \quad v(E_1) = 0
\]

\[
p(E_2) = \frac{12 \times \binom{40}{2}}{\binom{52}{3}} \quad \text{and} \quad v(E_2) = 1
\]

\[
p(E_3) = \frac{12 \times \binom{40}{3}}{\binom{52}{3}} \quad \text{and} \quad v(E_3) = 2
\]

\[
p(E_4) = \frac{\binom{12}{3}}{\binom{52}{3}} \quad \text{and} \quad v(E_4) = 3
\]
Substitute these values into

\[ E.V. = p(E_1) \times v(E_1) + p(E_2) \times v(E_2) + p(E_3) \times v(E_3) + p(E_4) \times v(E_4) \approx 0.6923 \]