12. Vectors and the geometry of space

12.1 Three dimensional coordinate systems

A point in the plane has two coordinates

A point in space has three coordinates

Describe objects by equations & inequalities

1. \( z \geq 0 \) All points \((x, y, z)\) in 3D space with \( z \geq 0 \) : All points on or above the xy plane

2. \( x = 3 \) Vertical plane through \((3, 0, 0)\) on the x-axis

3. \( 0 \leq x \leq 1 \) cube
   \[ 0 \leq y \leq 1 \]
   \[ 0 \leq z \leq 1 \]

4. Distance between points
In 3 dimensions, the distance between points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ equals

$$d = \sqrt{(z_2-z_1)^2 + (y_2-y_1)^2 + (x_2-x_1)^2}$$

**Question:** Which equation describes points $(x, y, z)$ on the sphere with radius 2 centered at the origin?

We have

$$\sqrt{(z-0)^2 + (y-0)^2 + (x-0)^2} = 2$$

or

$$x^2 + y^2 + z^2 = 4$$

The equation of a sphere with radius $r > 0$ and center $(x_0, y_0, z_0)$ is

$$\boxed{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2} \quad (*)$$

(5) Find the center and the radius of the sphere given by the equation

$$2x^2 + 2y^2 + 2z^2 + x + y + z = 9$$

We need to rewrite this equation and make it look like $(*)$
We accomplish this by completing the square:

\[ 9 = 2x^2 + x + 2y^2 + y + 2z^2 + z \]

\[ = 2\left(x^2 + \frac{x}{2}\right) + 2\left(y^2 + \frac{y}{2}\right) + 2\left(z^2 + \frac{z}{2}\right) \]

\[ 9 + \frac{3}{8} = 2\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + 2\left(y^2 + \frac{y}{2} + \frac{1}{16}\right) + 2\left(z^2 + \frac{z}{2} + \frac{1}{16}\right) \]

(Note that \(2 \cdot \frac{1}{16} + 2 \cdot \frac{1}{16} + 2 \cdot \frac{1}{16} = \frac{3}{8}\))

\[ 9 + \frac{3}{8} = 2\left(x + \frac{1}{4}\right)^2 + 2\left(y + \frac{1}{4}\right)^2 + 2\left(z + \frac{1}{4}\right)^2 \]

\[ \frac{75}{16} = \frac{9}{2} + \frac{3}{16} = (x + \frac{1}{4})^2 + (y + \frac{1}{4})^2 + (z + \frac{1}{4})^2 \]

This is the same as \((\star)\) if we choose \((x_0, y_0, z_0) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\) and \(r = \sqrt{75}/4\).

\(b\) Describe the set of all points equidistant to the origin and the point \((1, 2, 0)\):

Distance between \((x, y, z)\) and \((0, 0, 0)\):

\[ \sqrt{x^2 + y^2 + z^2} \]

Distance between \((x, y, z)\) and \((1, 2, 0)\):

\[ \sqrt{x^2 + (y-2)^2 + z^2} \]

We set the two equal (a.k.a. "equidistant")

\[ \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + (y-2)^2 + z^2} \]

\[ x^2 + y^2 + z^2 = x^2 + (y-2)^2 + z^2 \]

\[ y^2 = (y-2)^2 \]

\[ y^2 = y^2 - 4y + 4 \]
We get rid of $y^2$ so that

$0 = -4y + 4 \quad \Rightarrow \quad y = 1$

All points $(x, y, z)$ for which $y = 1$ lie on a

vertical plane
12.2 Vectors

Vectors are "Arrows" in 3D space or line segments with a distinguished direction.

They are used to model quantities which have a magnitude and a direction, such as velocity, acceleration or force. Quantities which have a magnitude but no direction are called scalars such as time, temperature, speed.

Two vectors with the same length & direction are considered equivalent.

In order to do algebra with vectors we need to describe them by coordinates.

Move the tail of the vector into the origin preserving length & direction.

We write $\vec{AB} = (x_0, y_0)$. $x_0, y_0$ are called the components of the vector $\vec{AB}$.