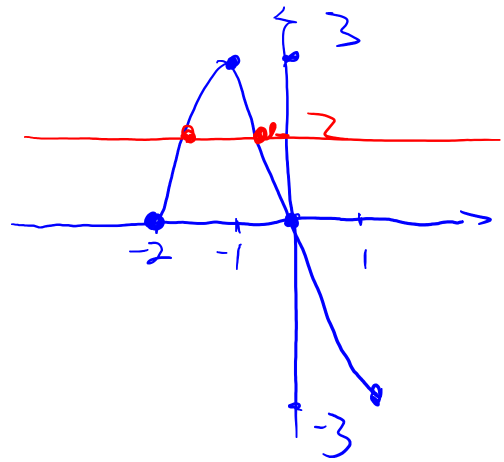
Practice Final, Sec18

Multiple Choice Problems.

1. Suppose $f(x)$ is a continuous function with values given by the table below.

x	-2	-1	0	1
f(x)	0	3	0	-3

Which of the following statement is correct?



A $f(x) = 2$ has a root $c \in (-1, 0)$.

B $f(x) = 2$ has a root $c \in (0, 1)$.

C $f(x) = 4$ has a root $c \in (-1, 0)$.

D $f(x) = 4$ has a root $c \in (-2, 1)$.

E None of the above

2. Suppose you are estimating the root of $x^3 = 5x - 1$ using Newton's method. If you use $x_1 = 2$, find the exact value of x_2

$f(x) = x^3 - 5x + 1 = 0$

$f'(x) = 3x^2 - 5$. $\quad f(2) = 2^3 - 5 \cdot 2 + 1 = -1$, $f'(2) = 3 \cdot 2^2 - 5 = 7$

$X_2 = X_1 - \frac{f(X_1)}{f'(X_1)} = 2 - \frac{-1}{7} = 2 + \frac{1}{7}$.

A $x_2 = 2 - \frac{1}{7}$

B $x_2 = 2 + \frac{1}{7}$

C $x_2 = 8 - \frac{8}{9}$

D $x_2 = 8 + \frac{8}{9}$

E $x_2 = 5 + \frac{1}{7}$

3. Evaluate the limit:

$$\lim_{x \to 3} \frac{x+2}{x(x-3)}$$

$\lim_{x \to 3^+} \frac{x+2}{x \cdot (x-3)} = \frac{5}{3 \cdot 0^+} = +\infty$

$\#$

$\lim_{x \to 3^-} \frac{x+2}{x \cdot (x-3)} = \frac{5}{3 \cdot 0^-} = -\infty$

A $+\infty$

B $-\infty$

C $\frac{5}{3}$

D $-\frac{5}{3}$

E The limit does not exist.

4. Find the horizontal aysmptote(s) of the following function:

$$f(x) = \frac{x-2}{3x+5}$$

$\lim_{x \to \infty} \frac{x-2}{3x+5} = \lim_{x \to \infty} \frac{x}{3x} = \frac{1}{3}$.

$y = \frac{1}{3}$.

A $x = \frac{1}{3}$

B $y = \frac{1}{3}$

C $x = -\frac{5}{3}$

D $y = 2$

E $y = -\frac{2}{5}$

5. Compute the limit:

$$\lim_{h\to 0} \frac{\frac{1}{h+2} - \frac{1}{2}}{h}$$

A $+\infty$

B $\frac{1}{2}$

C $\frac{1}{4}$

D $-\frac{1}{4}$

E 0

$$= \lim_{h\to 0} \frac{\frac{2-h-2}{(h+2)\cdot 2}}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{(h+2)\cdot 2\cdot \cancel{h}}$$

$$= \frac{-1}{2\cdot 2} = -\frac{1}{4}.$$

6. Find the limit:

$$\lim_{x\to 0} \frac{\sin(2x)}{3x}$$

A $\frac{2}{3}$

B $\frac{3}{2}$

C 0

D ∞

E Does not exist.

$$= \lim_{x\to 0} \cancel{\frac{\sin(2x)}{2x}} \cdot \frac{2x}{3x}$$

$$= \lim_{x\to 0} \frac{2\cancel{x}}{3\cancel{x}}$$

7. Suppose $\int_0^2 f(x)\,dx = -4$, $\int_0^5 f(x)\,dx = 6$. Find $\int_2^5 f(x)dx$ and the average of $f(x)$ over $[2,5]$

A $\int_2^5 f(x)dx = 2$, average of f is $\frac{2}{3}$

B $\int_2^5 f(x)dx = 10$, average of f is $\frac{10}{3}$

C $\int_2^5 f(x)dx = -10$, average of f is $-\frac{10}{3}$

D $\int_2^5 f(x)dx = -2$, average of f is $-\frac{2}{3}$

E $\int_2^5 f(x)dx = 10$, average of f is $\frac{10}{5}$

$$= \int_2^0 f\,dx + \int_0^5 f\,dx$$

$$= -\int_0^2 f\,dx + \int_0^5 f\,dx$$

$$= -(-4) + 6 = 10.$$

$$\text{Ave} = \frac{1}{5-2}\int_2^5 f\,dx = \frac{10}{3}$$

8. Evaluate

$$\int_{-\pi}^{\pi} \underbrace{\sin x \cdot \sqrt{\cos x + 2}}\,dx$$

odd function $f(-x) = -f(x)$.

A $\frac{4}{3}$

B 0

C $-\frac{4}{3}$

D $-\frac{2}{3}$

E 2

9. Evaluate the sum

$$\sum_{i=1}^{20} \frac{4-i}{2}$$

$= \sum_{i=1}^{20} 2 - \frac{i}{2}$

$= \sum_{i=1}^{20} 2 - \sum_{i=1}^{20} \frac{i}{2}$

$= 20\cdot 2 - \frac{1}{2}\cdot \frac{20\cdot 21}{2}$

$= 40 - \frac{20\cdot 21}{4}$.

A $40 - \frac{20\times 21}{2}$

B $40 - \frac{20\times 21}{4}$

C $20 - \frac{20\times 21}{2}$

D $20 - \frac{20\times 21}{4}$

E $\frac{20\times 21}{2}$

10. Evaluate the integral

$$\int \sqrt[3]{2x-8}\, \mathrm{d}x$$

$u = 2x-8$, $du = 2\cdot dx$

$= \int \sqrt[3]{u} \cdot \frac{du}{2}$

$= \frac{1}{2} \frac{1}{\frac{1}{3}+1} u^{\frac{1}{3}+1} = \frac{1}{2}\cdot\frac{3}{4}\cdot u^{\frac{4}{3}} = \frac{3}{8}(2x-8)^{\frac{4}{3}}$

A $\frac{3}{4}x^{\frac{4}{3}} + C$

B $\frac{3}{8}(2x-8)^{\frac{4}{3}} + C$

C $\frac{3}{4}(2x-8)^{\frac{4}{3}} + C$

D $\frac{3}{8}x^{\frac{4}{3}} + C$

E $\frac{1}{3}(2x-8)^{\frac{3}{2}} + C$

11. Find the average value of $f(x) = 2x+3$ on $[-1, 2]$;

$\frac{1}{2-(-1)} \int_{-1}^{2} f(x)\, dx = \frac{1}{3} \cdot \int_{-1}^{2} 2x+3\, dx$

$= \frac{1}{3} \cdot (x^2+3x)\Big|_{-1}^{2} = \frac{1}{3}\cdot(2^2+6) - \frac{1}{3}((-1)^2-3)$

$= \frac{1}{3}\cdot 10 - \frac{1}{3}\cdot(-2) = \frac{12}{3} = 4$.

A 4

B 12

C $\frac{8}{3}$

D -4

E 8

12. Sovle the initial value problem if

$$y' = \sin(\frac{x}{3}), \quad y(0) = 4$$

$y = \int \sin(\frac{x}{3}) dx$ $\quad u = \frac{x}{3}$, $du = \frac{dx}{3}$.

$= \int \sin u \cdot 3 du$

$= (-\cos u)\cdot 3 = -3\cos(\frac{x}{3}) + C$

$4 = y(0) = -3\cos 0 + C = -3 + C \Rightarrow C = 7$

A $-3\cos(\frac{x}{3}) + 1$

B $-\cos(\frac{x}{3}) + 7$

C $-3\cos(\frac{x}{3}) + 7$

D $-\frac{1}{3}\cos(\frac{x}{3}) + 1$

E $-3\sin(\frac{x}{3}) + 4$

Standard Response Problems.

1. Calculate the first and second order derivatives of $f(x) = x \sin x$. And find the equation of the tangent line to the curve $y = f(x)$ at $x = 0$

$$f'(x) = x' \cdot \sin x + x \cdot (\sin x)' = \boxed{1 \cdot \sin x + x \cdot \cos x.}$$

$$f''(x) = \cos x + x' \cdot \cos x + x \cdot \cos x'$$

$$= \cos x + \cos x + x \cdot (-\sin x) = \boxed{2\cos x - x \cdot \sin x.}$$

$$f'(0) = \sin 0 + 0 \cdot \cos 0 = 0.$$

Tangent line: $y - 0 = 0(x - 0) \Rightarrow \boxed{y = 0}$
at $(0, 0)$

2. Find the derivatives of

$$f(x) = \frac{\cos(x^2)}{\sqrt{x}}$$

$$f'(x) = \frac{(\cos(x^2))' \sqrt{x} - \cos(x^2) \cdot (\sqrt{x})'}{(\sqrt{x})^2}$$

$$= \boxed{\frac{\sin(x^2) \cdot 2x \sqrt{x} - \cos(x^2) \cdot \frac{1}{2\sqrt{x}}}{x}.}$$

$$= \frac{\sin(x^2) \cdot 4x - \cos(x^2)}{2x \cdot \sqrt{x}}.$$

3. Suppose that y and x satisfy the implicit equation

$$xy^3 + xy = 20$$

(a) Find $\frac{dy}{dx}$ $\quad (xy^3)' + (xy)' = (20)' = 0$

$x' \cdot y^3 + x \cdot (y^3)' + x' \cdot y + x \cdot y' = 0$

$y^3 + x \cdot 3y^2 \cdot y' + y + x \cdot y' = 0$

$(3xy^2 + x)y' = -y^3 - y$, $\boxed{\frac{dy}{dx} = \frac{-y^3 - y}{3xy^2 + x}}$

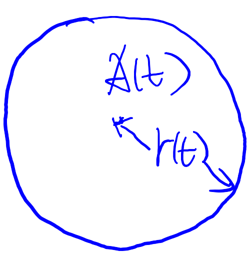
(b) Use your answer in part (a) to find the equation of the tangent line to the curve $xy^3 + xy = 20$ at the point $(10, 1)$.

$x = 10$, $y = 1$

$\frac{dy}{dx} = \frac{-1^3 - 1}{3 \cdot 10 \cdot 1^2 + 10} = \frac{-2}{40} = \frac{-1}{20}$.

$\boxed{y - 1 = -\frac{1}{20}(x - 10)}$

4. If the radius of a circular ink blot is growing at a rate of 3 cm/min. How fast (in cm^2/min) is the area of the blot growing when the radius is 10 cm?



$A(t) = \pi \cdot r^2(t)$.

$r' = 3$

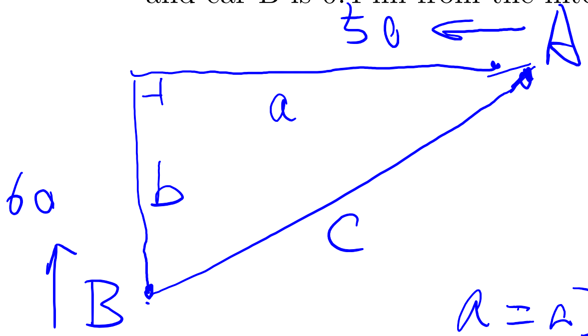
$r = 10$

$A'(t) = \pi \cdot 2r(t) \cdot r'(t)$.

$= \pi \cdot 2 \cdot 10 \cdot 3$

$= \boxed{60\pi \ \text{cm}^2/\text{min}}$

5. Car A is traveling west at 50 mi/h and car B is traveling north at 60mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?



$$a^2+b^2=c^2$$

$$2a\cdot a'+2b\cdot b'=2c\cdot c'$$

$$a=.3,\quad a'=50,\quad b=.4,\quad b'=60$$

$$c^2=.3^2+.4^2=.09+.16=.25$$

$$c=\sqrt{.25}=.5.$$

$$2\cdot .3\cdot 50+2\cdot .4\cdot 60=2\cdot .5\cdot c'$$

$$30+48=1\cdot c'$$

$$\boxed{c'=78.\ \text{mi/h}}$$

6. Find the absolute maximum and minimum of $f(x) = -x^3 + 3x$ on $[-1, 2]$.

$$f'(x)=-3x^2+3=0$$

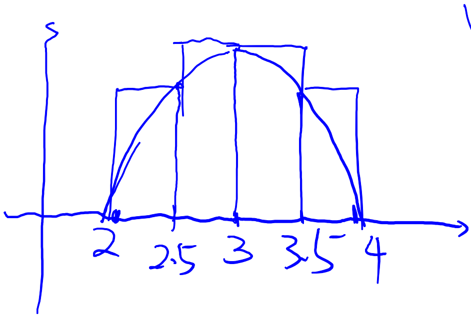
$$-3x^2=-3 \Rightarrow x^2=1 \Rightarrow x=1 \text{ or } -1.$$

x	$f(x)$
-1	$-(-1)^3+3\cdot(-1)=2$
1	$-1^3+3\cdot 1=2$
2	$-2^3+3\cdot 2=-2$

$\boxed{\text{abs max}=2}$ at $x=1$

$\boxed{\text{abs min}=-2}$ at $x=-1, 2.$

7. A particle moves with velocity $v(t) = -t^2 + 6t - 8$, $0 \le t \le 6$. Sketch the graph of $v(t)$ on $[2, 4]$. USE **FOUR RECTANGLES OF EQUAL WIDTH** to find the overestimate of the displacement of the particle traveled from $t = 2$ to $t = 4$.



$$\text{width} = \frac{4-2}{4} = \frac{1}{2}.$$

$$\text{Overestimate} = \frac{1}{2}\cdot\left[f(2.5) + f(3) + f(3) + f(3.5)\right]$$

$$= \boxed{\frac{1}{2}\left[-2.5^2+6\cdot 2.5-8+2(-3^2+6\cdot 3-8)-3.5^2+6\cdot 3.5-8\right]}$$

8. (S16) Suppose $f(x) = x^4 - 6x^2 - 3$.

(a) Identify the intervals over which $f(x)$ is increasing and decreasing, and all values of x where $f(x)$ attains its local maximum or minimum.

$$f'(x) = 4x^3 - 12x = 4x\cdot(x^2-3) = 4x\cdot(x+\sqrt{3})\cdot(x-\sqrt{3})$$

critical numbers: $x=0$, $x=-\sqrt{3}$, $x=\sqrt{3}$.

x		$-\sqrt{3}$		0		$\sqrt{3}$	
f'	$-$		$+$		$-$		$+$

increasing: $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

decreasing $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$

(b) Identify the intervals over which $f(x)$ is concave up and down, and all values of x where $f(x)$ has an inflection point.

$$f''(x) = (4x^3-12x)' = 12x^2-12 = 12(x^2-1) = 12(x+1)(x-1)$$

$f''(x) = 0 \Rightarrow x=-1$, $x=1$. inflection points

x		-1		1	
f''	$+$		$-$		$+$

concave up: $(-\infty, -1) \cup (1, \infty)$

concave down $(-1, 1)$.

9. Calculate the integral

$$\int \frac{x^2}{\sqrt{3+x^3}}\,dx$$

$u = 3 + x^3$

$du = 3\boxed{x^2 \cdot dx}$

$$= \int \frac{\frac{1}{3}du}{\sqrt{u}}$$

$$= \int \frac{1}{3} \cdot u^{-\frac{1}{2}} du = \frac{1}{3} \cdot 2u^{\frac{1}{2}}$$

$$= \boxed{\frac{2}{3} \cdot (3+x^3)^{\frac{1}{2}}}$$

10. Calculate the integral $\int_0^{\pi/4} \tan x \cdot \sec x + 2x\; dx$

$$= (\sec x + x^2)\Big|_0^{\frac{\pi}{4}}$$

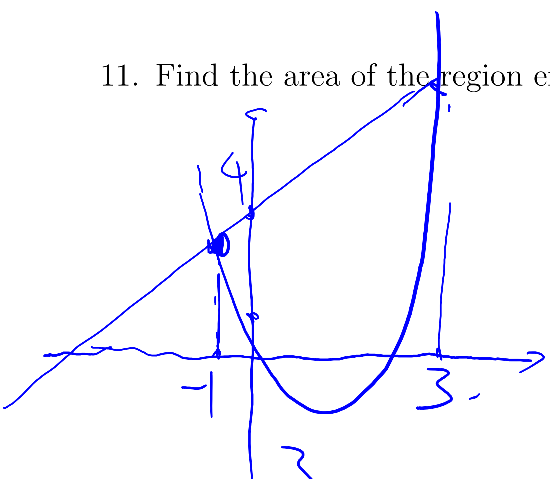
$$\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$= \sec\frac{\pi}{4} + \left(\frac{\pi}{4}\right)^2 - (\sec 0 + 0^2).$$

$$\sec 0 = \frac{1}{\cos 0} = 1.$$

$$= \boxed{\sqrt{2} + \left(\frac{\pi}{4}\right)^2 - 1.}$$

11. Find the area of the region enclosed by the graphs of the equations $y = x + 4$ and $y = x^2 - x + 1$.



$$y = x + 4 = x^2 - x + 1$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3,\; x = -1.$$

$$\text{Area} = \int_{-1}^{3} x + 4 - (x^2 - x + 1)\; dx = \int_{-1}^{3} x + 3 - x^2\; dx.$$

$$= \frac{1}{2}x^2 + 3x - \frac{1}{3}x^3 \Big|_{-1}^{3} = \boxed{\frac{9}{2} + 9 - \frac{1}{3} \cdot 27 - \left(\frac{1}{2} - 3 + \frac{1}{3}\right).}$$