Practice Mid 1, Sec18

**Q1**[*Sec1.4, Average rate of change/Average velocity, see also Q9*] Let $f(x) = \cos x + 2$. Compute the average rate of change of $f(x)$ on the interval $[0, \frac{\pi}{2}]$.
**Solution:** Average rate of change:

$$A.R.o.C. = \frac{f(\frac{\pi}{2}) - f(0)}{\frac{\pi}{2} - 0} = \frac{(\cos(\frac{\pi}{2}) + 2) - (\cos 0 + 2)}{\frac{\pi}{2} - 0} = \frac{(0+2) - (1+2)}{\frac{\pi}{2}} = \frac{-1}{\frac{\pi}{2}} = -\frac{2}{\pi} \tag{1}$$

**Q2**[*Sec1.5/1.6, Limit and Limit Laws*] Evaluate the following limits

**(a)Direct plug in-type**

Suppose $\lim_{x\to 4} f(x) = 2, \lim_{x\to 4} g(x) = 3.$ Find $\lim_{x\to 4} \frac{xf(x)+2}{f(x) - \sqrt{g(x)}}$

**Solution:**

$$\lim_{x\to 4} \frac{xf(x)+2}{f(x) - \sqrt{g(x)}} = \frac{\lim_{x\to 4} x \cdot \lim_{x\to 4} f(x) + 2}{\lim_{x\to 4} f(x) - \sqrt{\lim_{x\to 4} g(x)}} = \frac{4\cdot 2+2}{2-\sqrt{3}} = \frac{10}{2-\sqrt{3}} \tag{2}$$

**(b)$\frac{1}{0}$-type/One-sided limits**

$$\lim_{x\to 0^+} \frac{x-3}{x^2(x+5)}$$

**Solution:** $x \to 0^+ \implies x - 3 \to 0 - 3 = -3 < 0$(negative), $x^2 > 0$(positive), $x + 5 \to 0 + 5 = 5 > 0$(positive).

$$\frac{x-3}{x^2(x+5)} \sim \frac{negative}{positive \times postive} \sim negative, \implies \frac{x-3}{x^2(x+5)} \to \frac{0-3}{0^2\cdot(0+5)} = \frac{-3}{0\cdot 5} \to -\infty$$

$$\lim_{x\to 0^+} \frac{x-3}{x^2(x+5)} = -\infty$$

**(c)Absolute value**

$$\lim_{x\to 1^-} \frac{|x-1|}{x-1}$$

**Solution:** $\frac{0}{0}$-type. We need to cancel out the 'zero terms' then plug in $x = 1$. Before that, we need to remove the abstract value $|\cdot|$ first.

As $x \to 1^-$, $x < 1 \implies x - 1 < 0 \implies |x-1| = -\big(x-1\big)$. Then

$$\lim_{x\to 1^-} \frac{|x-1|}{x-1} = \lim_{x\to 1^-} \frac{-\big(x-1\big)}{x-1} = \lim_{x\to 1^-} -1 = -1$$

**(d)Cancellation-type**

$$\lim_{x\to -2} \frac{x^2-4}{x+2}$$

**Solution:**

$$\lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2} \frac{(x+2)(x-2)}{x+2}, \quad \textit{Cancel out the zero term } x+2 \tag{3}$$

$$= \lim_{x\to -2} x - 2 = (-2) - 2 = -4. \quad \textit{Plug in } x = -2 \tag{4}$$

**Q3**[*Sec1.8, Domain of continuity*] Use interval notation to indicate where $f(x)$ is continuous.

**(a)**

$$f(x) = \frac{x^2 - 3x + 1}{x - 3}. \quad \text{Choose from below}$$

**A**. $(-\infty, +\infty)$; **B**. $(-\infty, 3) \cup (3, +\infty)$; **C**. $(-\infty, 1) \cup (1, +\infty)$; **D**. $(-\infty, 1) \cup (1, 3) \cup (3, +\infty)$.

**Solution:** $f(x)$ is continuous everywhere in its domain. The domain of $f(x)$ is all those $x$ such that $f(x)$ is computable(meaningful/finite number). The only point not in $f$'s domain is $x = 3$, which makes the denominator zero. Therefore, $f(x)$ is continuous everywhere except $x = 3$. Option B.

**(b)**

$$f(x) = \sqrt{x+1}. \quad \text{Choose from below}$$

**A**. $(-\infty, +\infty)$; **B**. $(-\infty, -1]$; **C**. $[-1, +\infty)$; **D**. $(1, +\infty)$. **Solution:** Similar to part (a), $f(x)$ is continuous everywhere in its domain. The expression under square root has to be nonnegative, i.e., $x + 1 \geq 0 \Longrightarrow x \geq -1 \Longrightarrow x \in [-1, +\infty)$. Option C.

**(c)**

$$f(x) = \frac{(x^2 - 3x + 1)\sqrt{x+1}}{x - 3}. \quad \text{Use (a,b) to indicate the intervals of continuous for (c)}$$

**Solution:** The function contains both expression in (a) and (b). Therefore, the domain where $f(x)$ where it is continuous should satisfy both (a) and (b). Combine part (a) and part (b), we have the answer $[-1, 3) \cup (3, +\infty)$.

**Q4**[*Sec1.8, Piecewise function*] For what value of $k$ will $f(x)$ be continuous for all values of $x$ ?

$$f(x) = \begin{cases} \frac{x^2 - 3k}{x - 3}, & x \leq 2 \\ 8x - k, & x > 2 \end{cases}$$

Options: **A**. $k = 2$; **B**. $k = 3$; **C**. $k = 4$; **D**. $k = 5$.

**Solution:** $f(x)$ is a piecewise function which might have a break at the connecting point $x = 2$. The strategy is simply to plug $x = 2$ into the first and second expression of $f$. Then set them equal and solve for $k$.

Plug $x = 2$ into $\frac{x^2 - 3k}{x - 3}$, we get $\frac{2^2 - 3k}{2 - 3} = \frac{4 - 3k}{-1} = -(4 - 3k) = 3k - 4$.
Plug $x = 2$ into $8x - k$, we get $8x - k = 8 \cdot 2 - k = 16 - k$.
Set them equal: $3k - 4 = 16 - k \Longrightarrow 4k = 20 \Longrightarrow k = 5.$

The reason why these three steps give us the $k$ such that $f$ is continuous is as follows: $f(x)$ is continuous at $x = 2$ if and only if

$$(*) \quad f(2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

It graphically means that the left part of the curve and the right part of the curve are connected at $x = 2$. In the piecewise expression of $f(x)$, it is $\leq$ in the first part. Therefore,

$$f(2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 3k}{x - 3} = \frac{4 - 3k}{-1} = 3k - 4$$

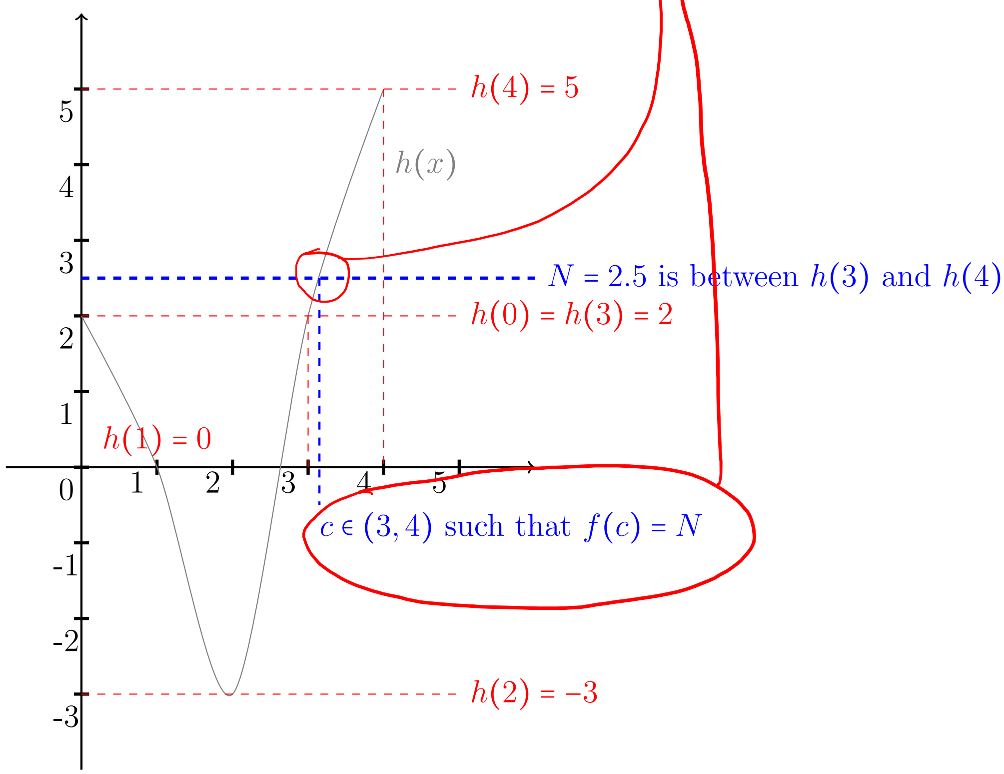
Similarly, we have

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} 8x - k = 8 \cdot 2 - k = 16 - k$$

Now due to $(*)$, it is enough to let $3k - 4 = 16 - k$ and solve for $k$.

**Q5**[*Sec1.8, Intermediate Value Theorem(IVT)* ] Suppose function $h(x)$ is continuous on $[0,4]$. Suppose $h(0) = 2, h(1) = 0, h(2) = -3, h(3) = 2, h(4) = 5$. For what value of $N$, the must be a $c \in (3,4)$ such that $h(c) = N$? Options: **A**. $N = 0.5$; **B**. $N = 0$; **C**. $N = -2$; **D**. $N = 2.5$.

**Intermediate Value Theorem(IVT)**: If $f$ is continuous on $[a,b]$, $f(a) \neq f(b)$, and $N$ is between $f(a)$ and $f(b)$ then there exists $c \in (a,b)$ that satisfies $f(c) = N$.



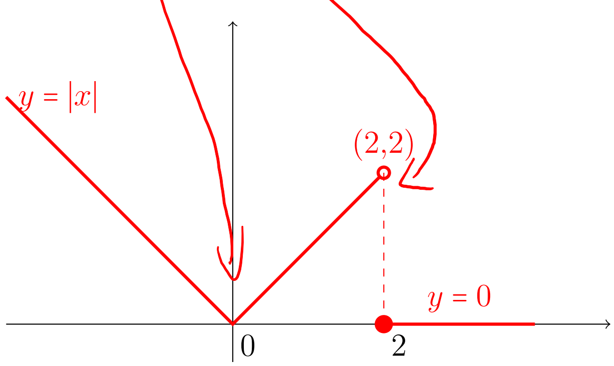

**Q6**[*Sec2.1/2.2, derivative at given point*] Select all true statements about the function $f(x) = \begin{cases} |x|, & x < 2 \\ 0, & x \geq 2 \end{cases}$

**(False) I** $f(x)$ is differentiable at $x = 0$. No

**(False) II** $f(x)$ is continuous at $x = 2$ No.

**(True)III** $\lim_{x\to 0} f(x)$ exists

Yes

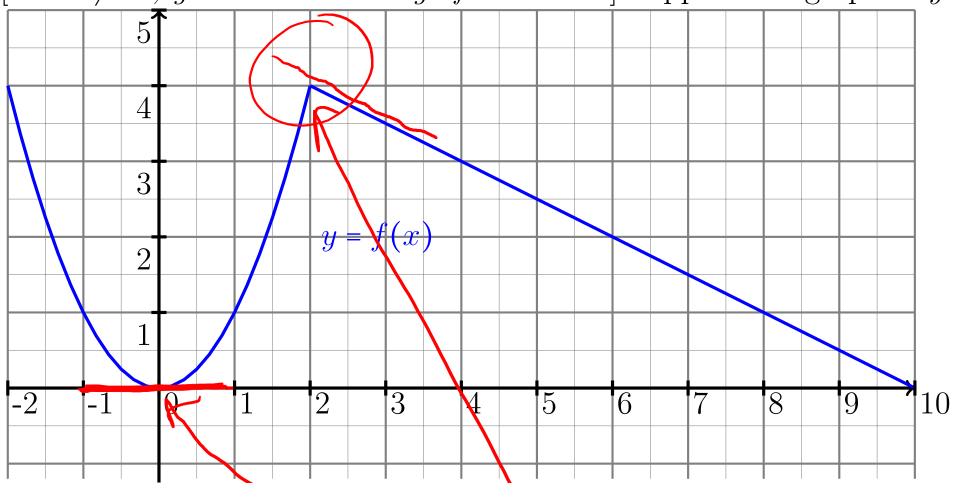



**Solution:**From the above graph, $f(x)$ has a jump at $x = 2$ (the left and right parts are not connected), therefore, $f(x)$ is not continuous at $x = 2$. $f(x)$ has a sharp turn at $x = 0$, the left line has slope $-1$ and the right line has slope $+1$, therefore, $f(x)$ is not differentiable at $x = 0$. Also we can read the limits of $f$ from the graph directly:

$$\lim_{0} f(x) = 0, \quad \lim_{2^-} f(x) = 2, \quad \lim_{2^+} f(x) = 0$$

Therefore, $\lim_{x\to 0} f(x)$ exists and $\lim_{x\to 2} f(x)$ does not exist.

**Q7**[*Sec2.1/2.2, geometric meaning of derivative*] Suppose the graph of $y = f(x)$ is given as follows:




Answer the following questions based on the above graph:

1. Find $f(0)$ and $f'(0)$. Find the equation of the tangent line of $y = f(x)$ at $(0, f(0))$.

   **Solution:** $f(0) = 0$ and $f'(0) = 0$. The tangent line at $(0, 0)$ is the horizontal axis, $y = 0$.

2. Is $f(x)$ continuous at $x = 2$? Yes. Is $f(x)$ differentiable at $x = 2$? No. The left and right tangent lines are not the same. Find

$$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h}$$

   **Solution:** The above expression is the (right) derivative of the function at $x = 2$. It is equal to the slope of the tangent line of the curve at $x = 2$ from the right hand side. It is enough to compute the slope of the straight line on $[2, 10]$ from its picture,i.e.:

$$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \text{slope} = \frac{f(10) - f(2)}{10 - 2} = \frac{0 - 4}{10 - 2} = -\frac{1}{2}$$

□

3. Find $f(6)$ and $f'(6)$. Find the equation of the tangent line of $y = f(x)$ at $(6, f(6))$.

   **Solution:** Read the data from the picture directly. $f(6) = 2$ and

   $f'(6) =$ the slope of the tangent line at $x = 6$ =the slope we got in the previous part=$-\frac{1}{2}$.

   Equation of the tangent line=Equation of the straight line itself (point-slope formula):

$$y - f(6) = \text{slope}\,(x - 6) \tag{5}$$

$$\Longrightarrow y - 2 = -\frac{1}{2}(x - 6) \tag{6}$$

$$\Longrightarrow y = -\frac{1}{2}(x - 6) + 2 = -\frac{1}{2}x + 3 + 2 = -\frac{1}{2}x + 5 \tag{7}$$

$$\Longrightarrow y = -\frac{1}{2}x + 5 \tag{8}$$

**Q8**[*Sec2.1/2.2, definition of derivative*] Let $f(x) = \frac{1}{x+1}$

**(a)** ***[Derivative as a limit]*** Use the definition of the derivative to find $f'(x)$. (Your calculation must include computing a limit.)

**Solution:**

$$f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h} = \lim_{h\to 0}\frac{\frac{x+1}{(x+h+1)(x+1)}-\frac{x+h+1}{(x+1)(x+h+1)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{(x+1)-(x+h+1)}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{-h}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h\to 0}\frac{-h}{(x+h+1)(x+1)h}$$

Fix $x$, plug $h=0$ in

$$= \lim_{h\to 0}\frac{-1}{(x+h+1)(x+1)}$$

$$= \frac{-1}{(x+0+1)(x+1)} = \frac{-1}{(x+1)^2}$$

**(b)** ***[Evaluating the derivative function at given point]*** Find $f'(2)$

**Solution:** By part (a), $f'(x) = \frac{-1}{(x+1)^2}$. Plug $x=2$ in, we have $f'(2) = \frac{-1}{(2+1)^2} = -\frac{1}{9}$.

**(c)** ***[Point-slope formula for the tangent line]*** Use part (b) to find an equation of a tangent line of $f(x)$ at $x=2$.

**Solution:** Slope$=f'(2) = -\frac{1}{9}$. Point: $(2, f(2))$, where $f(2) = \frac{1}{2+1} = \frac{1}{3}$. By Point-Slope formula, the equation of the tangent line at $x=2$ is given by:

$$y - \frac{1}{3} = (-\frac{1}{9})(x-2) \Longleftrightarrow y = (-\frac{1}{9})(x-2) + \frac{1}{3}$$

**Q8***[*Sec2.1/2.2, definition of derivative*] Use the definition of the derivative to find $g'(1)$ for $g(x) = 2\sqrt{x}$.
**Solution:**

$$g'(1) = \lim_{h\to 0}\frac{g(1+h)-g(1)}{h} = \lim_{h\to 0}\frac{2\sqrt{1+h}-2\sqrt{1}}{h} = \lim_{h\to 0}\frac{2\left(\sqrt{1+h}-1\right)}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$$

$$= \lim_{h\to 0}\frac{2\left(\sqrt{1+h}-1\right)\left(\sqrt{1+h}+1\right)}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0}\frac{2\left((\sqrt{1+h})^2-1^2\right)}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0}\frac{2(1+h-1)}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0}\frac{2\cancel{h}}{\cancel{h}\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0}\frac{2}{\sqrt{1+h}+1} = \frac{2}{\sqrt{1+0}+1} = 1$$

**Q9**[*Sec2.3/2.4/2.5, Differentiation Formulas/Laws*] Find the derivatives of the following functions. Do not need to simplify.

**(a)[Linear Rule+Power functions ]**

$$T(x) = 2\sqrt{x} - \frac{1}{2\sqrt{x}}$$

**Solution:**

$$\begin{aligned} T'(x) = \left(2\sqrt{x} - \frac{1}{2\sqrt{x}}\right)' &= \left(2x^{\frac{1}{2}}\right)' - \left(\frac{1}{2}x^{-\frac{1}{2}}\right)' \\ &= 2 \cdot \frac{1}{2}x^{\frac{1}{2}-1} - \frac{1}{2} \cdot (-\frac{1}{2})x^{-\frac{1}{2}-1} \\ &= x^{-\frac{1}{2}} + \frac{1}{4}x^{-3/2} \end{aligned}$$

**(b)[Product Rule+Power functions ]**

$$g(t) = \left(-1 + 2t\right)\left(\sin t + 2\right)$$

**Solution:**

$$\begin{aligned} g'(t) &= (-1 + 2t)'\left(\sin t + 2\right) + \left(-1 + 2t\right)(\sin t + 2)' \\ &= (0 + 2)\left(\sin t + 2\right) + \left(-1 + 2t\right)(\cos t + 0) \\ &= 2\left(\sin t + 2\right) + \left(-1 + 2t\right)\cos t \end{aligned}$$

**(c)[Trig functions+Chain Rule ]**

$$y = \sin(x^2)$$

**Solution:** Outer function: $\sin(\blacksquare)$; Inner function: $x^2$.

$$\begin{aligned} &outer' = (\sin(\blacksquare))' = \cos(\blacksquare) \to (\text{plug inner } x^2 \text{ in}) \to \cos(x^2); \\ &inner' = (x^2)' = 2x \\ &y' = (\sin(x^2))' = outer'(inner) \cdot inner' = \cos(x^2) \cdot (2x) \end{aligned}$$

**(c*)[Trig functions+Chain Rule ]**

$$y = \sin^2(x)$$

**Solution:** Outer function: $\blacksquare^2$; Inner function: $\sin x$.

$$\begin{aligned} &outer' = (\blacksquare^2)' = 2\blacksquare \to (\text{plug inner } \sin x \text{ in}) \to 2\sin x; \\ &inner' = (\sin x)' = \cos x \\ &y' = (\sin^2(x))' = outer'(inner) \cdot inner' = 2\sin x \cdot \cos x \end{aligned}$$

**(d)[Quotient Rule+Trig functions+Chain Rule ]**

$$f(t) = \frac{3t}{\tan(t^2-1)}$$

**Solution:** Apply quotient rule first with Numerator: $3t$; Denominator: $\tan(t^2-1)$.

$$\begin{aligned} f'(t) = \left(\frac{3t}{\tan(t^2-1)}\right)' &= \frac{(numerator)' \cdot denominator - numerator \cdot (denominator)'}{(denominator)^2} \\ &= \frac{(3t)' \cdot \tan(t^2-1) - 3t \cdot (\tan(t^2-1))'}{(\tan(t^2-1))^2} \\ &= \frac{3 \cdot \tan(t^2-1) - 3t \cdot (\tan(t^2-1))'}{(\tan(t^2-1))^2} \end{aligned}$$

To compute $(\tan(t^2-1))'$, we need chain rule with Outer function $\tan(\blacksquare)$ and Inner function $t^2-1$.

$$\begin{aligned} &outer' = (\tan(\blacksquare))' = \sec^2(\blacksquare) \to (\text{plug inner } t^2-1 \text{ in}) \to \sec^2(t^2-1); \\ &inner' = (t^2-1)' = 2t - 0 = 2t \\ &(\tan(t^2-1))' = outer'(inner) \cdot inner' = \sec^2(t^2-1) \cdot (2t) \end{aligned}$$

Plug $(\tan(t^2-1))'$ back to the quotient rule, we have

$$\boxed{f'(t) = \left(\frac{3t}{\tan(t^2-1)}\right)' = \frac{3 \cdot \tan(t^2-1) - 3t \cdot (\tan(t^2-1))'}{(\tan(t^2-1))^2} = \frac{3 \cdot \tan(t^2-1) - 3t \cdot \sec^2(t^2-1) \cdot (2t)}{(\tan(t^2-1))^2}}$$

**(e)[Trig functions+Double Chain Rule ]**

$$f(x) = 3\sec\big(\cos(1-2x)\big)$$

**Solution:** $f(x)$ is a composition of three functions: $3\sec(\blacksquare)$, $\cos(\blacksquare)$ and $1-2x$. We need to apply chain rule twice.

1st Chain rule: Outer function: $3\sec(\blacksquare)$; Inner function: $\cos(1-2x)$.

$$\begin{aligned} &outer' = (3\sec(\blacksquare))' = 3\sec(\blacksquare) \cdot \tan(\blacksquare) \\ &\qquad (\text{plug inner } \cos(1-2x) \text{ in}) \to 3\sec(\cos(1-2x)) \cdot \tan(\cos(1-2x)); \\ &inner' = (\cos(1-2x))' \\ (*): \quad &f'(x) = outer'(inner) \cdot inner' = 3\sec(\cos(1-2x)) \cdot \tan(\cos(1-2x)) \cdot (\cos(1-2x))' \end{aligned}$$

To compute $(\cos(1-2x))'$, we need to apply the second chain rule with Outer function: $\cos(\blacksquare)$; Inner function: $1-2x$.

$$\begin{aligned} &outer' = (\cos(\blacksquare))' = -\sin(\blacksquare) \to (\text{plug inner } 1-2x \text{ in}) \to -\sin(1-2x); \\ &inner' = (1-2x)' = 0 - (2x)' = -2 \\ (**): \quad &(\cos(1-2x))' = outer'(inner) \cdot inner' = -\sin(1-2x) \cdot (-2) \end{aligned}$$

Plug $(**)$ into $(*)$, we have

$$\begin{aligned} f'(x) &= 3\sec(\cos(1-2x)) \cdot \tan(\cos(1-2x)) \cdot (\cos(1-2x))' \\ &= 3\sec(\cos(1-2x)) \cdot \tan(\cos(1-2x)) \cdot \big(-\sin(1-2x) \cdot (-2)\big) \\ &= 3\sec(\cos(1-2x)) \cdot \tan(\cos(1-2x)) \cdot \sin(1-2x) \cdot \cancel{(-2)}\ +2. \\ &= 6\sec(\cos(1-2x)) \cdot \tan(\cos(1-2x)) \cdot \sin(1-2x). \end{aligned}$$

**Q9**[*Sec2.7, Rates of Change/Functions of motion*] A particle moves according to the law of motion $s(t) = t^3 - 5t^2 + 6t$, where $t$ is measured in seconds and $s$ in feet

**(a)[1.4, Average velocity ]** Find the average velocity over the interval $[0, 2]$.

**Solution:** Average velocity=Average rate of change of $s(t)$ over $[0, 2]$

$$\boxed{v_{ave} = \frac{s(2) - s(0)}{2 - 0}} = \frac{(2^3 - 5 \cdot 2^2 + 6 \cdot 2) - (0)}{2} = \frac{8 - 20 + 12}{2} = 0 \text{ ft/s}$$

**(b)[Velocity and position ]** Find the velocity $v(t)$ at time $t$.

**Solution:**

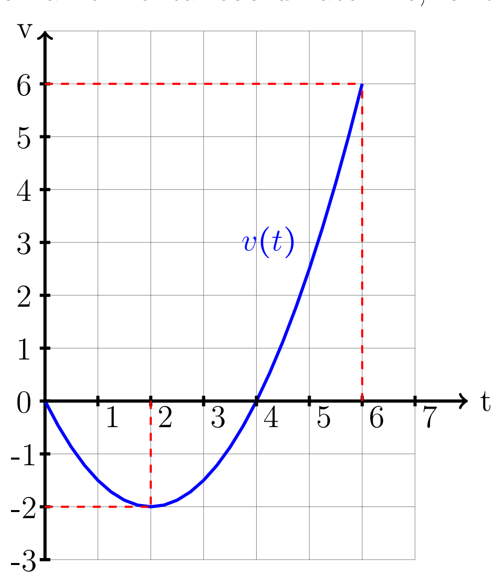
$$v(t) = s'(t) = (t^3 - 5t^2 + 6t)' = (t^3)' - (5t^2)' + (6t)' = 3t^2 - 5 \cdot 2t + 6 = 3t^2 - 10t + 6$$

**(c)[Acceleration and velocity ]** What is the acceleration $a(6)$ after 6 seconds?

**Solution:**

$$a(t) = v'(t) = (3t^2 - 10t + 6)' = (3t^2)' - (10t)' + (6)' = 3 \cdot 2t - 10 + 0 = 6t - 10$$
$$a(6) = 6 \cdot 6 - 10 = 26 \text{ ft/s}^2$$

**Q10**[*Sec2.7, Graph of the velocity*] The accompanying figure shows the velocity $v(t)$ of a particle moving on a horizontal coordinate line, for $t$ in the closed interval $[0, 6]$.




**Solution:**

**(a)** When does the particle move forward?

Move forward $\iff v > 0 \iff t \in (4, 6)$

**(b)** When does the particle slow down?

Slow down $\iff$ Speed $|v|$ drops $\iff t \in (2, 4)$

**(c)** When is the particle's acceleration positive?

acceleration positive $\iff a(t) = v'(t) > 0 \iff$ slope of the tangent line is positive/$v$ is increasing $\iff t \in (2, 6)$

**(d)** When does the particle move at its greatest speed in $[0, 6]$?

greatest speed $\iff$ highest or lowest point in the graph $\iff t = 6$ (greatest speed=6)

**Q11**[*Sec2.6, Implicit differentiation*] Consider the curve $y^2 + 2xy + x^3 = x$

**(a)** Find $\frac{dy}{dx}$ in terms of $x, y$. Apply Implicit differential rule to the equation $y^2 + 2xy + x^3 = x$.

$$\begin{aligned} (y^2 + 2xy + x^3)' &= (x)' \\ \Longrightarrow (y^2)' + (2xy)' + (x^3)' &= 1 \qquad (*) \\ \Longrightarrow 2y \cdot y' + 2y + 2xy' + 3x^2 &= 1 \qquad (**) \end{aligned}$$

From $(*)$ to $(**)$, we use the chain rule for $(y^2)'$ and product rule for $(2xy)'$, where

$$\begin{aligned} &\text{chain rule} : (y^2)' = 2y(x) \cdot y'(x) = 2yy' \\ &\text{product rule} : (2xy)' = (2x)' \cdot y(x) + 2x \cdot y'(x) = 2y + 2xy' \\ &(x^3)' = 3x^2 \end{aligned}$$

$(**)$: leave all the terms containing $y'$ on the left hand side and move all the rest terms to the right hand side of the equation, and then solve for $y'$:

$$\begin{aligned} &2y \cdot y' + 2y + 2x \cdot y' + 3x^2 = 1 \\ \Longrightarrow &2y \cdot y' + 2x \cdot y' = 1 - 2y - 3x^2 \\ \Longrightarrow &(2y + 2x) \cdot y' = 1 - 2y - 3x^2 \\ \Longrightarrow &\frac{dy}{dx} = y' = \frac{1 - 2y - 3x^2}{2y + 2x} \end{aligned}$$

**(b)** Find $\frac{dy}{dx}$ at $(1, -2)$ and find the slope of the tangent line of the curve at the point $(1, -2)$. Plug $(x, y) = (1, -2)$ into the expression in part (a), we have

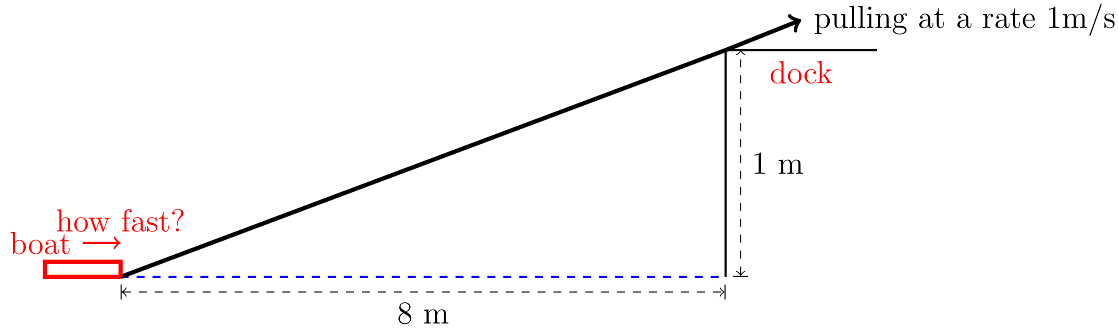
$$\begin{aligned} \frac{dy}{dx} = y' = \frac{1 - 2y - 3x^2}{2y + 2x} &= \frac{1 - 2 \times (-2) - 3 \times 1^2}{2 \times (-2) + 2 \times 1} \\ &= \frac{1 + 4 - 3}{-4 + 2} \\ &= \frac{2}{-2} \\ &= -1 \end{aligned}$$

**(c)** Find the equation of the tangent line of the curve at the point $(1, -2)$.

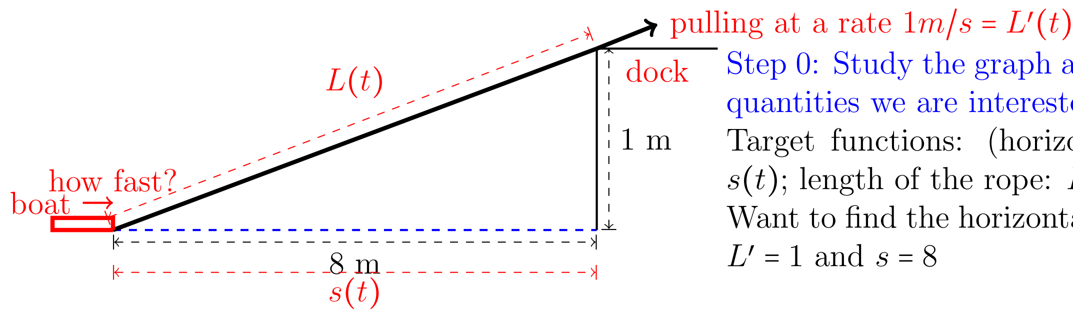
**Solution:** Slope=$-1$. Point (1,-2). The slope-point formula gives the formula for the tangent line:

$$y - (-2) = (-1)(x - 1) \Longleftrightarrow y = (-1)(x - 1) - 2$$

**Q12, *Sec2.8, Related Rates*** A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?




**Solution:**




Step 0: Study the graph and the problem, find all the quantities we are interested in.

Target functions: (horizontal) position of the boat $s(t)$; length of the rope: $L(t)$.

Want to find the horizontal velocity $v(t) = s'(t)$ given $L' = 1$ and $s = 8$

Step 1: set up the equation relating all the quantities we find in step 0.

Relation of $s$ and $L$ is given by Pythagorean theorem:

$$s^2(t) + 1^2 = L^2(t) \Longleftrightarrow s^2 + 1 = L^2 \qquad (*)$$

Step 2: take derivatives.

Take derivative both sides: $(s^2+1)' = (L^2)' \Longleftrightarrow (s^2)' + 1' = (L^2)'$. Since both $s = s(t)$ and $L = L(t)$ are functions of $t$, we need to apply chain rule to get $(s^2)' = 2s \cdot s', (L^2)' = 2L \cdot L'$.

Now we have

$$2s \cdot s' + 0 = 2L \cdot L' \Longleftrightarrow s \cdot s' = L \cdot L' \qquad (**).$$

Step 3: plug in. Give $s = 8$, from $(*)$ we can also figure out the corresponding $L$ as $8^2 + 1 = L^2 \Longleftrightarrow L^2 = 61 \Longrightarrow L = \sqrt{65}$.

Now we can plug $s = 8, L = \sqrt{65}, L' = 1$ into $(**)$ to solve for $s'$, i.e.,

$$8 \cdot s' = \sqrt{65} \cdot 1 \Longrightarrow s' = \frac{\sqrt{65}}{8} \Longrightarrow v(t) = s'(t) = \frac{\sqrt{65}}{8} \text{ m/s}$$