MTH 132, Practice Final, Sec39, Fall 2018

Multiple Choice Problems.

1. Suppose $f(x)$ is a continuous and differentiable function with values given by the table below.

x	-2	-1	0	1
f(x)	0	3	0	-3

Which of the following statements are correct?

I $f(x) = 2$ must have a root $c \in (-1, 0)$.

True. $f(-1) = 3$ and $f(0) = 0$. $N = 2$ is between 3 and 0. Therefore, by IVT, there must be a solution in $(-1, 0)$ to the equation $f(x) = 2$.

II $f(x) = 2$ must have a root $c \in (0, 1)$.

False. $f(0) = 0$ and $f(1) = -3$. $N = 2$ is NOT between 3 and 0. IVT does not apply.

III $f'(x) = 3$ must have a root $c \in (-1, 0)$.

False. Given interval $(-1, 0)$, by MVT, there is a $c \in (-1, 0)$ which solves $f'(x) = \frac{f(0)-f(-1)}{0-(-1)} = \frac{0-3}{1} = -3$.

IV $f'(x) = 0$ must have a root $c \in (-2, 0)$.

True. Given interval $(-2, 0)$, by MVT, there is a $c \in (-2, 0)$ which solves $f'(x) = \frac{f(0)-f(-2)}{0-(-2)} = \frac{0-0}{2} = 0$.

A. Only II; **B.** Only IV; **C. Only I and IV** ; **D.** Only II and III; **E.** None of the above.

2. Suppose you are estimating the root of $x^3 = 5x - 1$ using Newton's method. If you use $x_1 = 2$, find the exact value of x_2.

Rewrite $x^3 = 5x - 1$ as $x^3 - 5x + 1 = 0$ and let $f(x) = x^3 - 5x + 1$. Then $f'(x) = 3x^2 - 5$. Given $x_1 = 2$, by Newton's method, we have that $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2^3-5\cdot2+1}{3\cdot2^2-5} = 2 - \frac{-1}{7} = 2 + \frac{1}{7}$.

A $x_2 = 2 - \frac{1}{7}$.

B $x_2 = 2 + \frac{1}{7}$

C $x_2 = 8 - \frac{8}{9}$. **D** $x_2 = 8 + \frac{8}{9}$. **E** $x_2 = 5 + \frac{1}{7}$.

3. Consider the function

$$f(x) = \frac{2 - x^2}{x(x-3)}$$

Which of the following statements are correct?

I $f(x)$ has vertical asymptotes $y = 3$ and $y = 0$. False.

II $f(x)$ has vertical asymptotes $x = 0$ and $x = 3$. True. Vertical asymptotes: let $x(x-3) = 0$ and solve that $x = 0,\ x = 3$.

III $f(x)$ has a horizontal asymptote $y = -1$. True. Horizontal asymptotes: $f(x) = \frac{2-x^2}{x^2-3x}$. Then $\lim_{x\to\pm\infty} \frac{2-x^2}{x^2-3x} = \lim_{x\to\pm\infty} \frac{\cancel{2}-x^2}{x^2\cancel{-3x}} = \lim_{x\to\pm\infty} \frac{-x^2}{x^2} = \lim_{x\to\pm\infty} -1 = -1$. Therefore, $y = -1$ is the horizontal asymptote.

IV $f(x)$ has no horizontal asymptote. False by III.

V $f(x)$ has no slant asymptote. True by III. If $f(x)$ has horizontal asymptotes, then it has no slant asymptote.

A. Only II; **B.** Only II and IV; **C.** Only I,III and V; **D. Only II,III and V** ; **E.** None of the above.

4. Consider the function:

$$f(x) = x^3 + 6x$$

Which of the following statements are correct?

I $f(x)$ is an odd function.

True. $f(x)$ only has odd powers. $f(-x) = (-x)^3 + 6(-x) = -(x^3 + 6x)$.

II $f(x)$ is increasing for $x > 0$ and is decreasing on for $x < 0$.

False. $f'(x) = 3x^2 + 6 > 0$ for all x. $f(x)$ is increasing for all x and is nowhere decreasing.

III $f(x)$ is increasing on for all x. True.

IV $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

True. Concave Up=$f''(x)$ is positive and Concave Down=$f''(x)$ is negative.

$f''(x) = 6x$. $f''(x) > 0$ for $x > 0$, $f''(x) < 0$ for $x < 0$ and $f''(0) = 0$.

V $f(x)$ has no critical point and no inflection point.

False. $f(x)$ has no critical point by II. $f(x)$ has an inflection point at $x = 0$ by IV.

A. Only I, III, V; **B.** Only I,II, IV; **C. Only I,III, IV** ; **D.** Only III, IV; **E.** None of the above.

5. Compute the limit:

$$\lim_{x \to 5} \frac{\sqrt[3]{x} - \sqrt[3]{5}}{x - 5}$$

Let $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$. Then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. By the limit-definition of the derivative at $x = a$,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a}.$$

Therefore, the given limit

$$\lim_{x \to 5} \frac{\sqrt[3]{x} - \sqrt[3]{5}}{x - 5} = f'(5) = \frac{1}{3}5^{-\frac{2}{3}}.$$

A $+\infty$

B 0

C $\frac{1}{3}\sqrt[3]{5}$

D $\frac{1}{3}5^{-\frac{2}{3}}$

E Does not exist

6. Find the limit:

$$\lim_{x \to 0} 3x \cdot \sin(\frac{1}{2x})$$

$|\sin(\bullet)| \leq 1$, no matter what $\bullet$ it is inside. $3x \longrightarrow 0$ as $x \to 0$. Therefore, $3x \cdot \sin(\bullet) \longrightarrow 0$.

A $\frac{2}{3}$

B $\frac{3}{2}$

C 0

D ∞

E Does not exist.

7. For what value of k will $f(x)$ be continuous for all values of x?

$$f(x) = \begin{cases} \frac{x^2-3k}{x-3} & \text{if } \; x \leq 2; \\ 8x - k & \text{if } \; x > 2; \end{cases}$$

Plug $x = 2$ into the two expressions of $f(x)$ and set them equal. Then solve for k.

$$\frac{2^2 - 3k}{2-3} = 8 \cdot 2 - k \Longrightarrow 3k - 4 = 16 - k \Longrightarrow 4k = 20 \Longrightarrow k = 5$$

A. $k = 2$. B. $k = 3$. C. $k = 4$. **D.** $k = 5$. E. None of the above.

8. Evaluate

$$\int_{-\pi}^{\pi} \sin x \cdot \sqrt{\cos x + 2}\; dx$$

Method I. $f(x) = \sin x \cdot \sqrt{\cos x + 2}$ is an ODD function and is integrated on the interval $[-\pi, \pi]$ symmetric at the origin. Then $\int_{-\pi}^{\pi} \sin x \cdot \sqrt{\cos x + 2}\; dx = 0$.

Method II. U-sub. $u = \cos x + 2, du = -\sin x\, dx$. Then

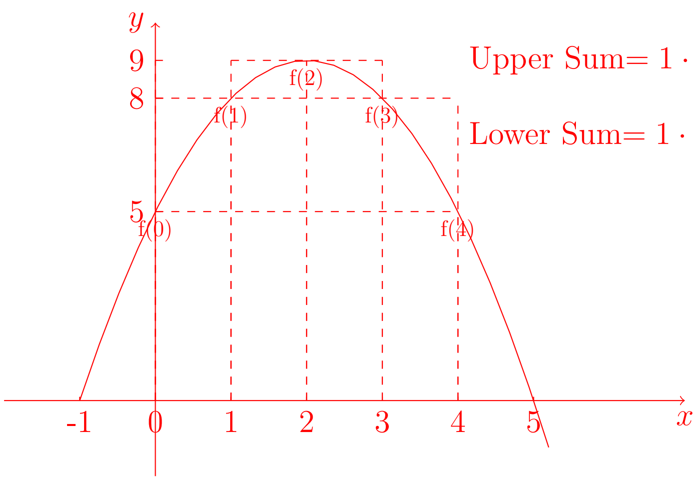
$$\int_{-\pi}^{\pi} \sin x \cdot \sqrt{\cos x + 2}\; dx = \int \sqrt{u} \cdot (-du) = -\frac{2}{3}u^{\frac{3}{2}} = -\frac{2}{3}(\cos x + 2)^{\frac{3}{2}}\Big|_{-\pi}^{\pi}$$

$$(Notice\ that\ \cos\pi = \cos(-\pi) = -1) \qquad = -\frac{2}{3}(\cos\pi + 2)^{\frac{3}{2}} + \frac{2}{3}(\cos(-\pi) + 2)^{\frac{3}{2}}$$

$$= -\frac{2}{3}(-1+2)^{\frac{3}{2}} + \frac{2}{3}(-1+2)^{\frac{3}{2}} = 0$$

A. $\frac{4}{3}$. **B.** 0. C. $-\frac{4}{3}$. D. $-\frac{2}{3}$. E 2.

9. Estimate the area under the graph of $f(x) = -x^2 + 4x + 5$ from $x = 0$ to $x = 4$ using the areas of 4. rectangles of equal width.



Upper Sum= $1 \cdot (f(1) + f(2) + f(2) + f(3)) = 8 + 9 + 9 + 8 = 34$

Lower Sum= $1 \cdot (f(0) + f(1) + f(3) + f(4)) = 5 + 8 + 8 + 5 = 26$

A The upper sum (overestimate) is 34 and the lower sum (underestimate) is 30.

B The upper sum (overestimate) is 34 and the lower sum (underestimate) is 26.

C The upper sum (overestimate) is 30 and the lower sum (underestimate) is 26.

D The upper sum (overestimate) is 17 and the lower sum (underestimate) is 13.

E None of the above.

10. A car is moving according to $s(t) = -t^2 + 4t + 5$. Which of the following statements are correct?
Velocity at time t, $v(t) = s'(t) = -2t + 4$. So $v(2) = -2 \cdot 2 + 4 = 0$.
The average velocity from $t = a$ to $t = b$,

$$v_{\text{ave}} = \frac{s(b) - s(a)}{b - a}$$

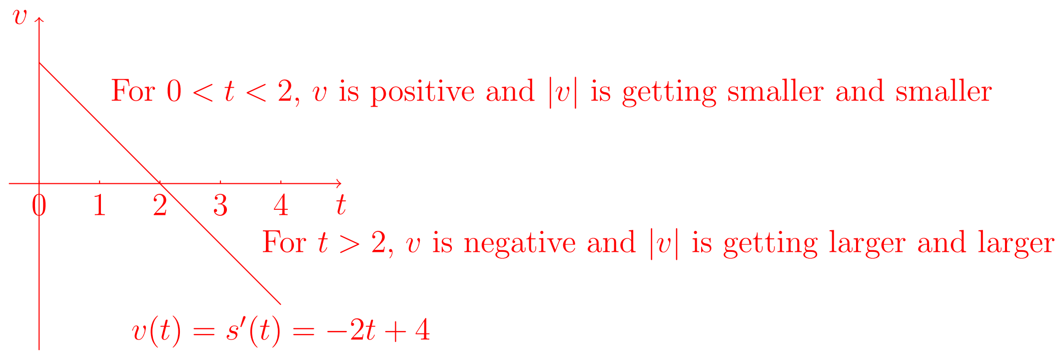
So

$$v_{\text{ave}} = \frac{s(2) - s(0)}{2 - 0} = \frac{-2^2 + 4 \cdot 2 + 5 - (0 + 0 + 5)}{2} = 4$$

I The velocity at $t = 2$ is 4.
II The velocity at $t = 2$ is 0.
III The average velocity from $t = 0$ to $t = 2$ is 4.5.
IV The average velocity from $t = 0$ to $t = 2$ is 5.

A. Only II; **B.** Only III; **C.** Only I and IV; **D.** Only II and III; **E.** None of the above.

11. A car is moving according to $s(t) = -t^2 + 4t + 5$. Which of the following statements are correct?
Slowing down means $|v|$ is decreasing; Speeding up means $|v|$ is increasing. Moving in positive direction means $v > 0$; Moving in negative direction means $v < 0$.



I The car is slowing down from $t = 0$ to $t = 2$.
II The car is speeding up from $t = 2$ to $t = 4$.
III The car is moving in positive direction from $t = 0$ to $t = 5$.
IV The car is moving in negative direction from $t = 2$ to $t = 5$.

A. Only I; **B.** Only III; **C. Only I,II and IV**; **D.** Only I and II; **E.** None of the above.

12. Find y if $y' = 2x\sin(x^2)$, $y(0) = 4$.
y is the integral (anti-derivative) of y': $y = \int 2x\sin(x^2)\,dx$.
U-sub, $u = x^2, du = 2xdx$. $y = \int 2x\sin(x^2)\,dx = \int \sin u\,du = -\cos u + C = -\cos(x^2) + C$.
Plug $x = 0, y = 4$ in and solve for C. $4 = -\cos 0 + C \Longrightarrow C = 5 \Longrightarrow y = -\cos(x^2) + 5$.

A $y = -2\cos(x) + 6$
B $y = -\cos(x^2) + 5$
C $y = \cos(x^2) + 3$
D $y = x^2\cos(\frac{x^3}{3}) + 4$
E $y = x^2\sin(\frac{x^3}{3}) + 4$

Standard Response Problems.

1. Find $\frac{dy}{dx}$ if

   (a) $y = x\tan(x^2)$ Direct computation by product rule and chain rule:

$$\begin{aligned}\frac{dy}{dx} =& (x)'\tan x^2 + x\cdot\left(\tan(x^2)\right)' \\ =& 1\cdot\tan x^2 + x\cdot\sec(x^2)\cdot\left(x^2\right)' \\ =& \tan x^2 + x\cdot\sec(x^2)\cdot(2x) \\ =& \tan x^2 + 2x^2\cdot\sec(x^2)\end{aligned}$$

   (b) $x^2 = 3y + \cos(y)$ Implicit differential rule:

$$\begin{aligned}(x^2)' = (3y+\cos(y))' \Longrightarrow & 2x = 3y' + (\cos(y))' \\ \Longrightarrow & 2x = 3y' - \sin(y)y', \quad \text{(then solve for } y') \\ \Longrightarrow & 2x = (3-\sin y)\ \cdot y' \\ \Longrightarrow & \frac{dy}{dx} = y' = \frac{2x}{3-\sin y}\end{aligned}$$

2. If the radius of a circular ink blot is growing at a rate of 3 cm/min. How fast (in cm^2/min) is the area of the blot growing when the radius is 10 cm?

   Related Rates: Area: $A(t)$; Radius: $r(t)$. $A = \pi r^2$. Take derivative with respect to t both sides of the equation:

$$A' = \left(\pi r^2\right)' = \pi\, 2r\ \cdot r', \quad \text{(Caution: Do not forget the } r' \text{ since } r = r(t) \text{ is a function of } t)$$

   Then plug $r = 10$ and $r' = 3$ in

$$A = \pi\, 2r\ \cdot r' = \pi\cdot 2\cdot 10\cdot 3 = 60\pi,\ (\text{cm}^2/\text{min})$$

3. Let

$$f(x) = \frac{1}{2x - x^2} \qquad \text{for} \quad x \in (0,2).$$

Let $f(0) = f(2) = 3$.

(a) Is $f(x)$ continuous on $[0,2]$?

No. $f(x)$ is not continuous at $x = 0$ and $x = 2$, since

$$\lim_{x\to 0^+} \frac{1}{2x - x^2} = \lim_{x\to 0^+} \frac{1}{x(2-x)} = \frac{1}{0^+ \cdot 2} = +\infty \neq f(0), \qquad \lim_{x\to 2^-} \frac{1}{x(2-x)} = \frac{1}{2\cdot 0^+} + \infty \neq f(2).$$

(b) Find the critical point and the local minimum of $f(x)$ in $(0,2)$.

Use Chain Rule or Quotient Rule to compute

$$f'(x) = \left((2x - x^2)^{-1}\right)' = (-1)(2x-x^2)^{-2} \cdot \left(2x - x^2\right)' = \frac{-1}{(2x-x^2)^2} \cdot (2 - 2x) = \frac{2x-2}{(2x-x^2)^2}$$

Set $f'(x) = 0$ and solve for x. $\frac{2x-2}{(2x-x^2)^2} = 0 \Longrightarrow 2x - 2 = 0 \Longrightarrow x = 1$ (Critical Point). It is also easy to check that $f'(x) > 0$ for $x \in (0,1)$ and $f'(x) < 0$ for $x \in (1,2)$ since

$$f'(0.5) = \frac{2\cdot 0.5 - 2}{(2\cdot 0.5 - 0.5^2)^2} < 0, \quad \text{and} \quad f'(1.5) = \frac{2\cdot 1.5 - 2}{(2\cdot 1.5 - 1.5^2)^2} > 0$$

$f(x)$ attains its local minimum at $x = 1$, the local minimum is $f(1) = \frac{1}{2\cdot 1 - 1^2} = 1$.

(c) Does $f(x)$ have absolute maximum or minimum in $[0,2]$.

$f(x)$ has the absolute minimum at at $x = 1$, which is $f(1) = \frac{1}{2\cdot 1 - 1^2} = 1$. $f(x)$ DOES NOT have the absolute maximum at in $[0,2]$ since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 2^-} f(x) = +\infty.$$

There is NO $x \in [0,2]$ such that $f(x)$ is greater than all the other values of $f(x)$.

4. Evaluate

$$\int \frac{x^2}{\sqrt{3 + x^3}}\, dx$$

U-sub: Let $u = 3 + x^3$. Then $du = 3x^2\, dx \Longrightarrow x^2 dx = \frac{1}{3} du$.

$$\begin{aligned}
\int \frac{x^2}{\sqrt{3+x^3}}\, dx = \int \frac{1}{\sqrt{3+x^3}}\, x^2 dx &= \int \frac{1}{\sqrt{u}}\, \frac{1}{3} du \\
&= \frac{1}{3} \int u^{-\frac{1}{2}}\, du \\
&= \frac{1}{3} \cdot \frac{1}{\frac{1}{2}} u^{\frac{1}{2}} \\
&= \frac{1}{3} \cdot 2\,(3 + x^3)^{\frac{1}{2}} + C
\end{aligned}$$

5. Suppose $f(x) = x^4 - 6x^2 - 3$.

(a) Identify the intervals over which $f(x)$ is increasing and decreasing, and all values of x where $f(x)$ attains its local maximum or minimum.

$f'(x) = 4x^3 - 12x = 4x(x^2-3) = 4x(x-\sqrt{3})(x+\sqrt{3})$. Solving $f'(x) = 0$ gives us all the critical numbers of f, which are $x = 0$,$x = \sqrt{3}$ and $x = -\sqrt{3}$.

$x = 0, x = \sqrt{3}, x = -\sqrt{3}$ break the whole interval into four sub-intervals: $(-\infty, -\sqrt{3})$,$(-\sqrt{3}, 0)$,$(0, \sqrt{3})$, and $(\sqrt{3}, \infty)$. We need to check whether f' is positive or negative on each interval by plugging numbers into $f'(x)$.

$f'(-2) < 0 \Longrightarrow f'(x) < 0$ for $x \in (-\infty, -\sqrt{3})$.

$f'(-1) > 0 \Longrightarrow f'(x) > 0$ for $x \in (-\sqrt{3}, 0)$.

$f'(1) < 0 \Longrightarrow f'(x) < 0$ for $x \in (0, \sqrt{3})$.

$f'(2) > 0 \Longrightarrow f'(x) > 0$ for $x \in (\sqrt{3}, \infty)$.

$f(x)$ is increasing on $(-\sqrt{3}, 0) \bigcup (\sqrt{3}, \infty)$. $f(x)$ is decreasing on $(-\infty, -\sqrt{3}) \bigcup (0, \sqrt{3})$. $f(x)$ attains local minimum at $x = -\sqrt{3}$ and $x = \sqrt{3}$. $f(x)$ attains local maximum at $x = 0$.

(b) Identify the intervals over which $f(x)$ is concave up and down, and all values of x where $f(x)$ has an inflection point.

$f''(x) = 12x^2 - 12 = 12(x^2-1) = 12(x-1)(x+1)$. Solving $f''(x) = 0$ gives us all the possible inflection points of f, which are $x = 1$, and $x = -1$.

$x = 1$, and $x = -1$ break the whole interval into three sub-intervals: $(-\infty, -1)$,$(-1, 1)$,$(1, \infty)$. We need to check whether f'' is positive or negative on each interval by plugging numbers into $f''(x)$.

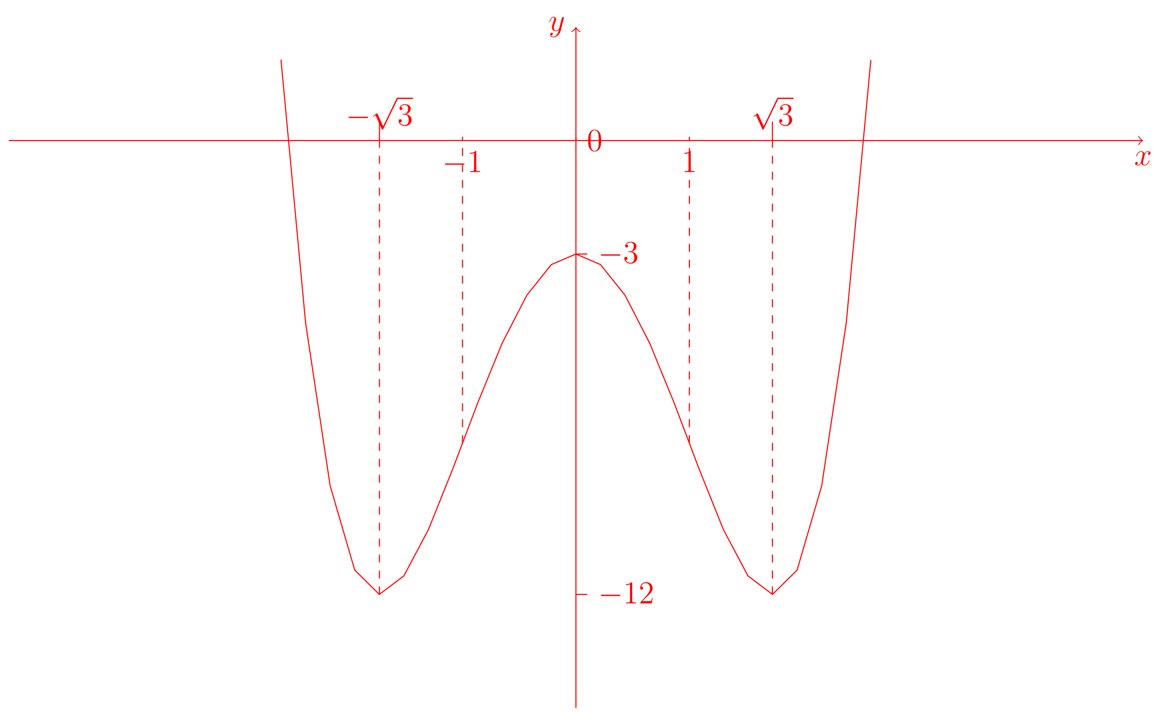
$f'(-2) > 0 \Longrightarrow f''(x) > 0$ for $x \in (-\infty, -1)$.

$f'(0) < 0 \Longrightarrow f''(x) < 0$ for $x \in (-1, 1)$.

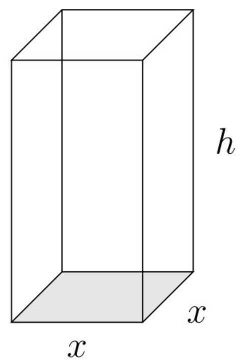
$f'(2) > 0 \Longrightarrow f''(x) > 0$ for $x \in (1, \infty)$.

$f(x)$ is concave up on $(-\infty, -1) \bigcup (1, \infty)$. $f(x)$ is concave down on $(-1, 1)$. $f(x)$ has inflection points at both $x = -1$ and $x = 1$.

(c) Sketch the graph of $y = f(x)$.



6. A box with square base and open top is to have a volume of 32 in^3. Find the dimensions of the box that minimizes the amount of material used.



Volume$= x^2 \cdot h = 32 \implies h = \frac{32}{x^2}$. We want to minimize the the amount of material used, i.e., the total surface area.

$$\text{Area} = x^2 + 4xh = x^2 + 4x\frac{32}{x^2} = x^2 + \frac{128}{x}, \quad x > 0$$

$$\frac{dA}{dx} = \left(x^2 + \frac{128}{x}\right)' = 2x - \frac{128}{x^2} = \frac{2x^3 - 128}{x^2}$$

Let $\frac{dA}{dx} = 0$ and solve for x. We have that $2x^3 - 128 = 0 \implies x^3 = 64 \implies x = 4$.

Notice that

$$\frac{dA}{dx} = \frac{2(x^3 - 64)}{x^2}$$

Plug in $x = 1$ and $x = 5$ to determine the signs of $\frac{dA}{dx}$. We have that $\frac{dA}{dx} < 0$ for $0 < x < 4$ and $\frac{dA}{dx} > 0$ for $x > 4$. Therefore, A attains its absolute minimum at $x = 4$.

The dimensions are $x = 4$ in and $h = \frac{32}{x^2} = 32/16 = 2$ in. The minimum material to be used is $A(4) = 4^2 + \frac{128}{4}$ in^2.

7. Find the average of the function

$$f(x) = \frac{\cos x}{\sin^2 x}$$

over the interval $[\frac{\pi}{4}, \frac{\pi}{2}]$.

First, compute the definite integral of the function over the interval:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} \cos x dx \qquad u = \sin x, du = \cos x\, dx$$

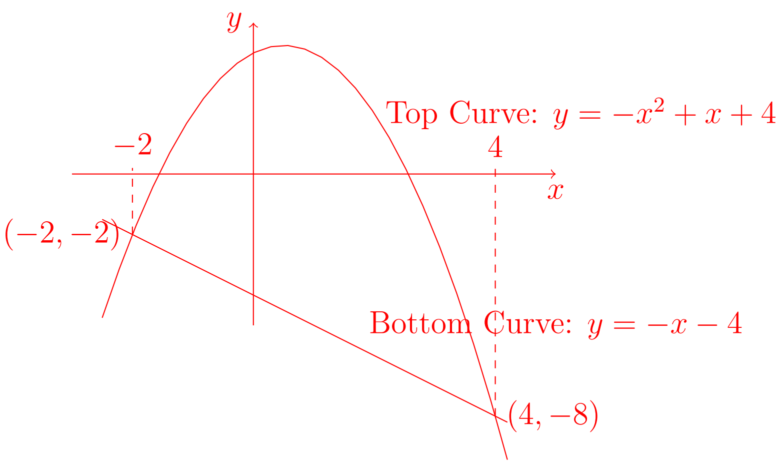
$$= \int \frac{1}{u^2}\, du = \int u^{-2}\, du = -u^{-1}$$

$$= -\frac{1}{\sin x}\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{1}{\sin\frac{\pi}{2}} + \frac{1}{\sin\frac{\pi}{4}} = -1 + \frac{1}{\frac{\sqrt{2}}{2}} = -1 + \sqrt{2}$$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{\frac{\pi}{2} - \frac{\pi}{4}} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx = \frac{1}{\frac{\pi}{4}}(-1 + \sqrt{2})$$

8. Find the area of the region enclosed by the graphs of the equations $y = -x - 4$ and $y = -x^2 + x + 4$.

Plot the two curves first:



Set $-x - 4 = -x^2 + x + 4$ and solve for the intersections:

$$0 = x^2 - 2x - 8 = (x-4)(x+2) = 0 \Longrightarrow x = 4,\ x = -2$$

Then the area of region equals:

$$\int_{-2}^{4} \text{Top} - \text{Bot}\, dx = \int_{-2}^{4} \left(-x^2 + x + 4 - (-x-4)\right)\, dx$$

$$= \int_{-2}^{4} -x^2 + 2x + 8\, dx$$

$$= -\frac{1}{3}x^3 + x^2 + 8x\Bigg|_{-2}^{4}$$

$$= -\frac{1}{3}4^3 + 4^2 + 8\cdot 4 - \left(-\frac{1}{3}(-2)^3 + (-2)^2 + 8\cdot(-2)\right)$$

$$= -\frac{64}{3} + 16 + 32 - \left(\frac{8}{3} + 4 - 16\right)$$

$$= 36$$