

## §3.2 Mean Value Theorem (MVT)

Mean Value Theorem (MVT): If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there exists  $c \in (a, b)$  that satisfies  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

key points: ① The requirements for  $f$ : continuous on the CLOSED interval  $[a, b]$  and differentiable on the OPEN interval  $(a, b)$ .

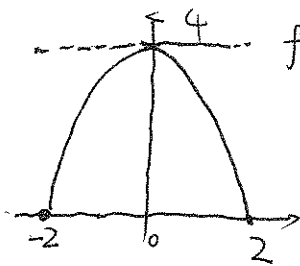
② Geometric meaning of MVT.

③ Find (Solve for)  $c$  in MVT.

e.g. (Baby version, Rolle's Theorem).

Consider the function  $f(x) = 4 - x^2$  on  $[-2, 2]$ . Sketch the graph of  $f$ .

Apply MVT to  $f$  with  $a = -2, b = 2$ . What does MVT tell you in the graph?



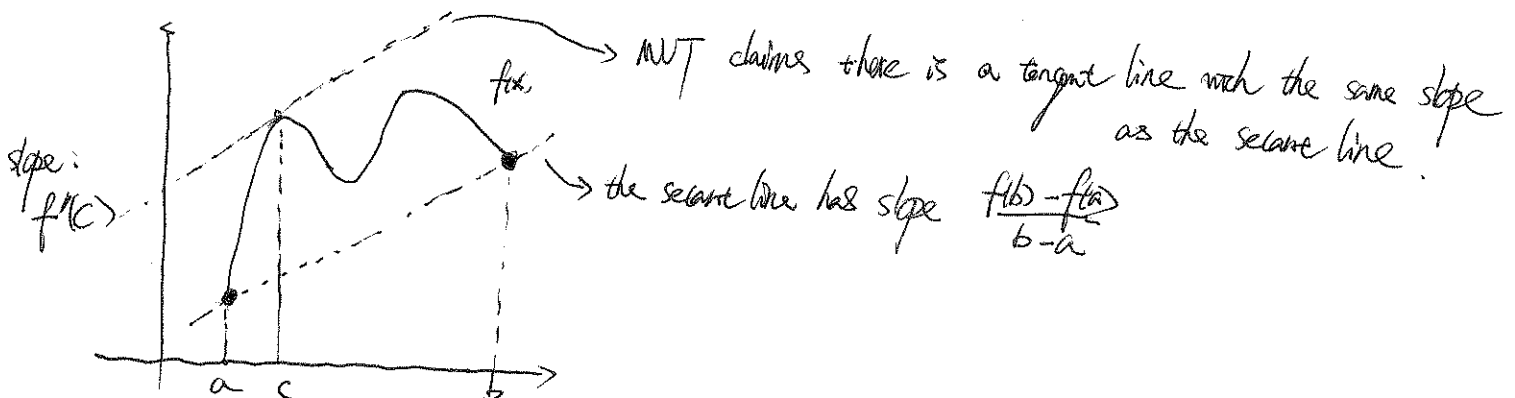
$$f(-2) = f(2) = 4 - 4 = 0.$$

$f$  is continuous and differentiable on  $[-2, 2]$ .

MVT claims there exists  $c \in [-2, 2]$  such that

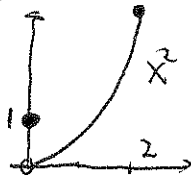
$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = 0. \text{ Actually, we know in this example } c = 0, \text{ since } f'(x) = (4 - x^2)' = -2x \Rightarrow f'(0) = 0.$$

Remark:  $\frac{f(b) - f(a)}{b - a}$  is the slope of the straight line passing through  $(a, f(a))$ ,  $(b, f(b))$ . (secant)



eg2. Can the MVT be applied to the following functions? why?

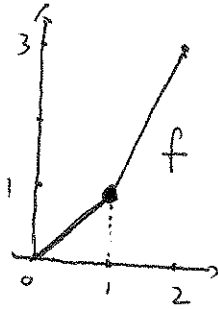
$$\textcircled{1} f(x) = \begin{cases} 1 & x=0 \\ x^2 & 0 < x \leq 2 \end{cases} \text{ on } [0, 2]$$



$f(x)$  has a break (jump) at  $x=0$ , i.e., not continuous at  $x=0$ , i.e., not continuous on  $[0, 2]$ . (closed interval).

No. MVT doesn't apply since  $f$  is not continuous on  $[0, 2]$

$$\textcircled{2} f(x) = \begin{cases} x & 0 \leq x < 1 \\ 2x-1 & 1 \leq x \leq 2 \end{cases} \text{ on } [0, 2]$$



$f$  has a sharp turn at  $x=1$ , therefore,  $f$  is not differentiable at  $x=1$ , i.e., not on  $(0, 2)$ .

No. MVT doesn't apply since  $f$  is not differentiable on  $(0, 2)$ .

eg3. Consider the function  $f(x) = x^2 - 4x$  on  $[1, 4]$ . Apply MVT to  $f$  on  $[1, 4]$  (F/B, MC)  
Find the number  $c$  which satisfies the conclusion of the theorem.

Solution:  $f(x) = x^2 - 4x$  on  $[1, 4]$ ,  $f$  is continuous on  $[1, 4]$  and differentiable on  $(1, 4)$ .

$a=1$ ,  $b=4$ . Conclusion: there is  $c \in (1, 4)$  such that  $f'(c) = \frac{f(4) - f(1)}{4 - 1}$ .

Goal: compute  $f'(x)$  and solve for  $c$  in the above equation.

$$f'(x) = (x^2 - 4x)' = 2x - 4 \Rightarrow f'(c) = 2c - 4$$

$$f(4) = 4^2 - 4 \cdot 4 = 0, \quad f(1) = 1^2 - 4 \cdot 1 = -3 \Rightarrow \boxed{2c - 4 = \frac{0 - (-3)}{4 - 1}}$$

$$2c - 4 = \frac{3}{3} = 1 \Rightarrow 2c = 5 \Rightarrow \boxed{c = \frac{5}{2}}$$

eg.4 Find the number  $c$  that satisfies the conclusion of MVT for  $f(x) = x^3 + x$  on  $(-1, 0)$

Solution:  $a = -1$ ,  $b = 0$ ,  $f(-1) = (-1)^3 + (-1) = -2$ ,  $f(0) = 0$ .  $\frac{f(0) - f(-1)}{0 - (-1)} = \frac{0 - (-2)}{0 - (-1)} = 2$

$$f'(x) = 3x^2 + 1 \Rightarrow f'(c) = \boxed{3c^2 + 1 = 2} = \frac{f(0) - f(-1)}{0 - (-1)}$$

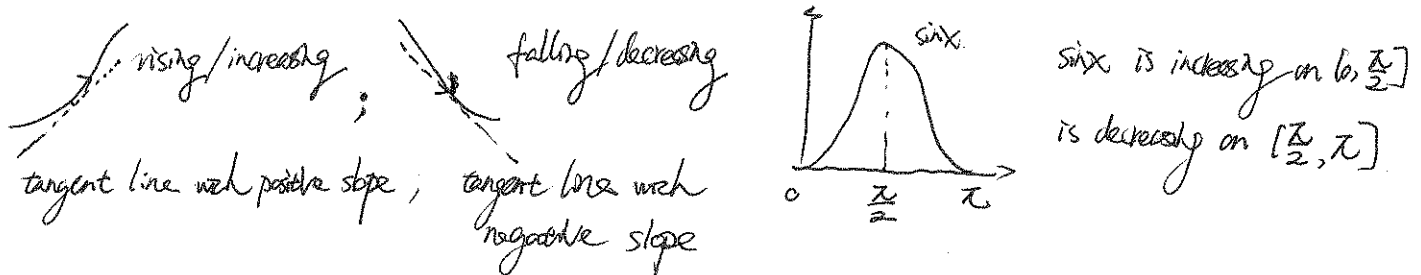
Solve for  $c$ :  $3c^2 = 1$ ,  $c^2 = \frac{1}{3} \Rightarrow c = \pm \frac{1}{\sqrt{3}} \Rightarrow c = -\frac{1}{\sqrt{3}} \in (-1, 0)$   
(the positive  $c = \frac{1}{\sqrt{3}}$  is discarded)

## ★ §3.3. Derivatives and Graphs

- Key points:
- ① Signs of  $f'(x)$  and monotonicity (increasing/decreasing) of  $f$ .
  - ② Signs of  $f''(x)$  and concavity (up/down) of  $f$ .
  - ③ First/Second Derivative Test for local maximum/minimum.
  - ④ Sketch the curve of  $f(x)$  via  $f'(x)$  and  $f''(x)$ . (signs).

Def:  $f(x)$  is increasing on  $[a, b]$  if the GRAPH IS RISING, i.e.,  $f(x_1) < f(x_2)$  for all  $x_1 < x_2$

$f(x)$  is decreasing on  $[a, b]$  if the GRAPH IS FALLING, i.e.,  $f(x_1) > f(x_2)$  for all  $x_1 < x_2$

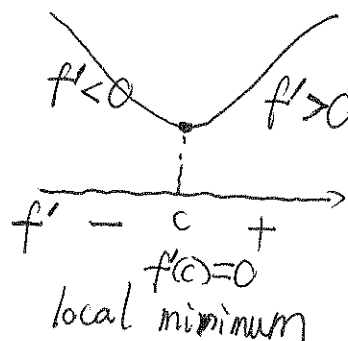
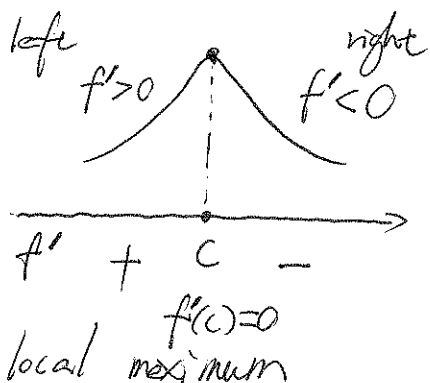


★ Theorem: Let  $f(x)$  be continuous on  $[a, b]$ .

- If  $f'(x) > 0$ , then  $f$  is increasing.
- If  $f'(x) < 0$ , then  $f$  is decreasing.

Sign of $f'$	+	-	0
$f$	increasing	decreasing	critical point

- First Derivative Test (for local extremum): ( $c$  is a critical point of  $f$ ),  $f'(c) = 0$ . If  $f'(x)$  has different signs on the left and right hand sides of  $c$ , then  $f(x)$  has a local extremum at  $x=c$ .

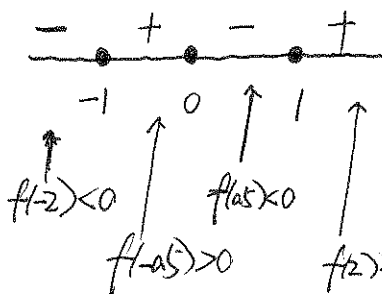


eg. 1 Suppose  $f(x) = x^4 - 2x^2 - 3$ . Find the intervals over which (FIB).  $f(x)$  is increasing and decreasing, and all values of  $x$ , where  $f(x)$  attains its local maximum or minimum.

Solution:  $f'(x) = (x^4 - 2x^2 - 3)' = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$

Remark: We factorize  $4x^3 - 4x = 4x(x+1)(x-1)$  since we want to determine  $f'$ 's signs.

signs of  $f'$



$f'$  has three zeros,  $-1, 0, 1$ , which break the ~~real~~ axis into four parts  $(-\infty, -1), (-1, 0), (0, 1), (1, +\infty)$ . Plug in some simple numbers in each part to determine the sign

$f' > 0 \Rightarrow f$  is increasing  $\Rightarrow [-1, 0] \cup [1, +\infty)$  ~~increasing~~  
 $f' < 0 \Rightarrow f$  is decreasing  $\Rightarrow (-\infty, -1] \cup [0, 1]$

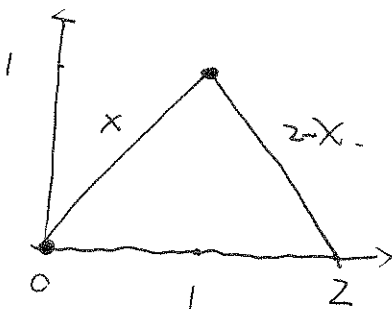
at  $x = -1$ , left  $-$ , right  $+$   $\cup$  ;  $f$  has local minimum at  $x = -1$ .

at  $x = 0$ , left  $+$ , right  $-$   $\cap$  :  $f$  has local maximum at  $x = 0$

at  $x = 1$ , left  $-$ , right  $+$   $\cup$  :  $f$  has local minimum at  $x = 1$ .

Remark: If the graph of  $f(x)$  can be determined directly, then use graph to find the intervals / local extrema of  $f$ .

eg. 2.  $f(x)$  on  $[0, 2]$  is defined as  $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \end{cases}$




$f(x)$  is increasing on  $[0, 1]$


decreasing on  $[1, 2]$

attains local (absolute) maximum at  $x = 1$


attains local minimum at  $x = 0$  and  $x = 2$ .

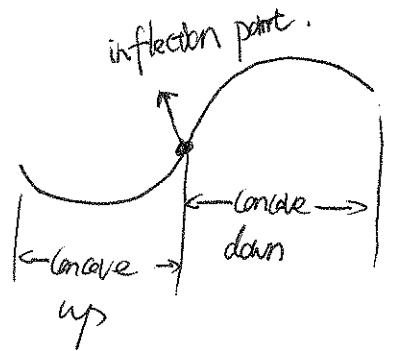
• Def. (Concavity):

•  $f$  is concave up if the graph is part of a smiling curve: 

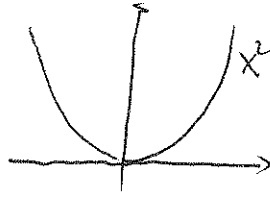
•  $f$  is concave down if the graph is part of a frowning curve: 

Concave up: 

Concave down: 

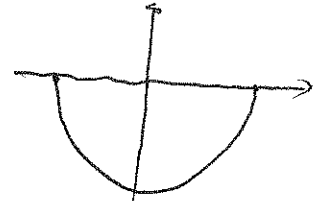


•  $y = x^2$  is concave up

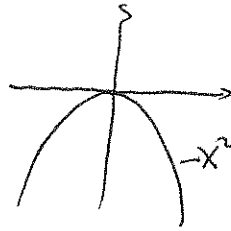


Lower semi circle:  $y = -\sqrt{1-x^2}$

is concave up

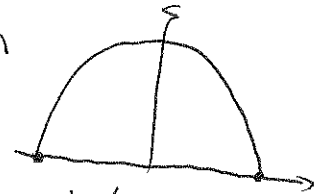


•  $y = -x^2$  is concave down



Upper semi circle  $y = \sqrt{1-x^2}$

is concave down



• Def:  $f(x)$  has an inflection point at  $x=C$  if  $f'(x)$  has a local extremum at  $C$ .

OR:  $C$  is an inflection point if  $f$  is concave up on one side of  $C$  and concave down on the other side

★ Theorem: •  $f''(x) > 0$  over  $(a,b)$ , then  $f(x)$  is concave up on  $(a,b)$

•  $f''(x) < 0$  over  $(a,b)$ , then  $f(x)$  is concave down on  $(a,b)$

•  $f''(c) = 0$  and  $f''(x)$  has different signs on the two sides of  $C$ , then  $C$  is an inflection point.

signs of  $f''$             +            -            0 (and changes signs)

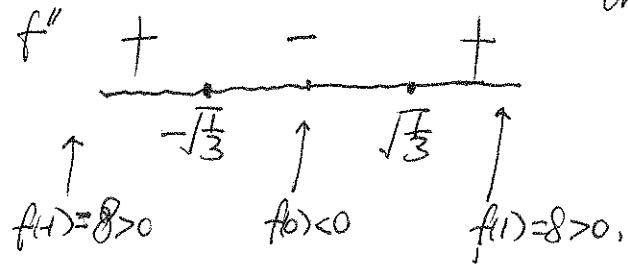
$f$             concave up    concave down    inflection point.

eg. 3. For  $f(x) = x^4 - 2x^2 - 3$  in eg. 1. Find where is  $f$  concave up/down and its inflection pts. ( $\star$  and then sketch the curve of  $y = f(x)$ .)

Solution:  $f'(x) = 4x^3 - 4x \Rightarrow f''(x) = 12x^2 - 4 = 4(3x^2 - 1) = 4(\sqrt{3}x + 1)(\sqrt{3}x - 1)$

or  $f''(x) = 12(x^2 - \frac{1}{3}) = 12(x + \frac{1}{\sqrt{3}})(x - \frac{1}{\sqrt{3}})$

$f''(\pm \frac{1}{\sqrt{3}}) = 0$

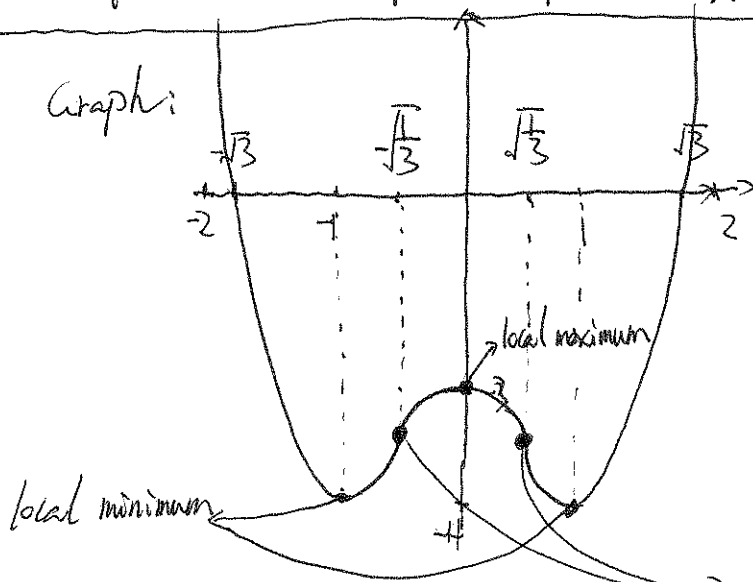


Concave up:  $f'' > 0 : (-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, +\infty)$

Concave down:  $f'' < 0 : (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

$f$  has two inflection points:  $x = -\frac{1}{\sqrt{3}}$  and  $x = \frac{1}{\sqrt{3}}$

Graph:



y intercept:  $f(0) = -3$

$\star$  x intercepts:  $f(x) = x^4 - 2x^2 - 3 = 0$

$\Leftrightarrow (x^2 - 3)(x^2 + 1) = 0$

$\Leftrightarrow (x + \sqrt{3})(x - \sqrt{3})(x^2 + 1) = 0$

x intercepts:  $x = \pm\sqrt{3}$

$f(\pm 1) = 1 - 2 - 3 = -4$

$f(\pm \frac{1}{\sqrt{3}}) = (\frac{1}{\sqrt{3}})^4 - 2(\frac{1}{\sqrt{3}})^2 - 3 = -3 - \frac{5}{9}$

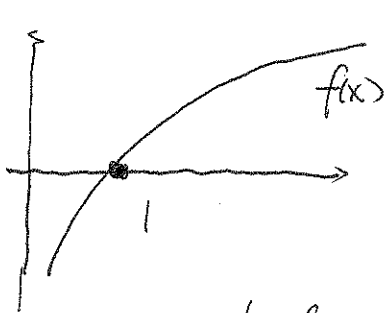
Second Derivative Test: Suppose  $f'(c) = 0$ .

- $f''(c) < 0$ , then  $f$  attains a local maximum at  $x=c$
- $f''(c) > 0$ , then  $f$  attains a local minimum at  $x=c$
- $f''(c) = 0$ , then the second derivative test is inconclusive.

Remark: In most cases, the first derivative test will be enough for local extrema.

More examples from actual exams.

eg. 4. The graph of  $f$  is given below. What can you say about  $f(1)$ ,  $f'(1)$ ,  $f''(1)$ ? (Compare the signs of them). the signs of



$$f(1) = 0, \text{ (x intercept is 1)}$$

$$f(x) \text{ is increasing (near 1)} \Rightarrow f'(1) > 0$$

$$f(x) \text{ is concave down} \Rightarrow f''(1) < 0$$

$$\text{Therefore, } f''(1) < f(1) < f'(1)$$

eg. 5 Find the inflection point (coordinates) for the function  $f(x) = \sin x - \cos x$  in  $[0, \pi]$ .

Solution: Inflection point  $\Rightarrow f'' = 0$ .

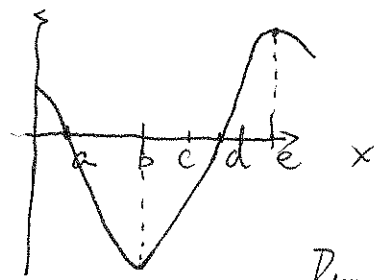
$$f'(x) = (\sin x - \cos x)' = (\sin x)' - (\cos x)' = \cos x - (-\sin x) = \cos x + \sin x$$

$$f''(x) = (\cos x)' + (\sin x)' = -\sin x + \cos x = 0$$

$$-\sin x + \cos x = 0 \Rightarrow x = \frac{\pi}{4}, \quad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 0$$

The inflection point at  $x = \frac{\pi}{4}$  is  $\left(\frac{\pi}{4}, 0\right)$ .

eg. 6. The graph of  $f'(x)$  is given. At what value of  $x$  does  $f(x)$  has a local maximum?



Caution: This is the graph of  $f'$ . NOT of  $f$

$x=e$  is not a local maximum.

Recall: local maximum  $\Rightarrow f'=0$  and  $f'$  changes signs.

From the graph,  $f'(a)=f'(b)=0$ .

✓  $x=a$ :  $f'$ :  $+$   $\rightarrow$   $-$  ~~graph~~  $\rightarrow$   $+$   $\rightarrow$   $-$  local maximum

$x=d$ :  $f'$ :  $-$   $\rightarrow$   $+$   $\rightarrow$   $-$   $\rightarrow$   $+$  local minimum

Hints for webwork:

ww 15:  $y = A \cdot x^{\frac{1}{4}} + B \cdot x^{-\frac{1}{4}}$  has an inflection point at  $(1, 6)$ . Find  $A, B$ .

Plug in  $x=1, y=6$ :  $6 = A \cdot 1^{\frac{1}{4}} + B \cdot 1^{-\frac{1}{4}} \Rightarrow 6 = A + B$ .

Find  $y''$ .  $y' = A \cdot \frac{1}{4} \cdot x^{-\frac{3}{4}} + B \cdot (-\frac{1}{4}) \cdot x^{-\frac{5}{4}}$

$$y'' = A \cdot \frac{1}{4} \cdot (-\frac{3}{4}) \cdot x^{-\frac{7}{4}} + B \cdot (-\frac{1}{4}) \cdot (-\frac{5}{4}) \cdot x^{-\frac{9}{4}}$$

$$= -A \cdot \frac{3}{16} \cdot x^{-\frac{7}{4}} + B \cdot \frac{5}{16} \cdot x^{-\frac{9}{4}}$$

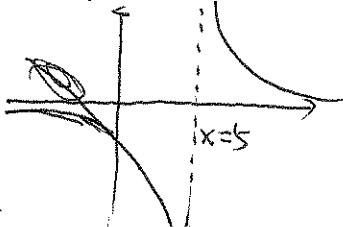
inflection point  $\Rightarrow x=1, y''=0 \Rightarrow 0 = -A \cdot \frac{3}{16} + B \cdot \frac{5}{16} \Rightarrow 0 = -3A + 5B$

$$\begin{cases} 6 = A + B \\ 0 = -3A + 5B \end{cases} \text{ solve for } A, B \Rightarrow A = \frac{5B}{3}, 6 = A + B = \frac{5B}{3} + B = \frac{8}{3}B$$

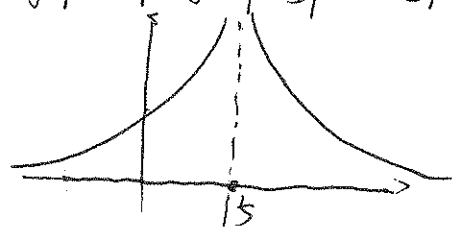
$$\Rightarrow B = \frac{18}{8}, A = \frac{5}{3} \cdot \frac{18}{8}$$

ww 8:

graph of  $y = \frac{1}{x-5}$



$\Rightarrow$  graph of  $y = \frac{1}{|x-5|} = \frac{1}{|x-5|} \Rightarrow$  increasing on  $(-\infty, 5)$



decreasing on  $(5, +\infty)$