

## §1.6 Calculating limits using the Limit Laws.

Key points: ★① (Linear) **Limit Laws** by general combination

★★② Limits by canceling zeros: Factoring technique.

③ Squeeze Theorem.

① Limit Laws:

• Sum/Difference:  $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$

• Constant multiple:  $\lim_{x \rightarrow a} [C \cdot f(x)] = C \cdot \lim_{x \rightarrow a} f(x)$ ,  $\lim_{x \rightarrow a} C = C$ .

• Product:  $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \left[ \lim_{x \rightarrow a} f(x) \right] \cdot \left[ \lim_{x \rightarrow a} g(x) \right]$

• Quotient:  $\lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$ .

• Power/Root:  $\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n$ ,  $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ .

• Constant/Polynomials:  $\lim_{x \rightarrow a} C = C$ ,  $\lim_{x \rightarrow a} x = a$ ,  $\lim_{x \rightarrow a} x^2 = a^2$ ,  $\lim_{x \rightarrow a} x^3 = a^3$ , ...

$\lim_{x \rightarrow a} \sqrt{x} = \sqrt{a}$  ( $a > 0$ ),  $\lim_{x \rightarrow a} \frac{1}{x} = \frac{1}{a}$  ( $a \neq 0$ ),  $\lim_{x \rightarrow a} \frac{1}{x^2} = \frac{1}{a^2}$ , ...

eg. Suppose  $\lim_{x \rightarrow 2} f(x) = -1$ ,  $\lim_{x \rightarrow 2} g(x) = 3$ . Compute the following limits.

•  $\lim_{x \rightarrow 2} \left[ 5 - \frac{x^3}{f(x) + g(x)} + 2 \cdot [f(x)]^2 \cdot \sqrt{g(x)} \right]$

$= \lim_{x \rightarrow 2} [5] - \lim_{x \rightarrow 2} \left[ \frac{x^3}{f(x) + g(x)} \right] + \lim_{x \rightarrow 2} \left[ 2 \cdot [f(x)]^2 \cdot \sqrt{g(x)} \right]$

$= 5 - \frac{\lim_{x \rightarrow 2} x^3}{\left[ \lim_{x \rightarrow 2} f(x) \right] + \left[ \lim_{x \rightarrow 2} g(x) \right]} + 2 \cdot \left[ \lim_{x \rightarrow 2} f(x) \right]^2 \cdot \sqrt{\lim_{x \rightarrow 2} g(x)}$

$= 5 - \frac{2^3}{-1 + 3} + 2 \cdot (-1)^2 \cdot \sqrt{3}$

$= 5 - 4 + 2 \cdot \sqrt{3} = \boxed{1 + 2\sqrt{3}} \quad *$

Rank: All the laws could be applied to one-sided limits.

eg. 2 (Direct plug in)

$$\bullet \lim_{x \rightarrow 0} \frac{x+1}{3x^2-5x+7} = \frac{0+1}{0+7} = \frac{1}{7}; \bullet \lim_{h \rightarrow 1} \frac{2h-h^2}{h+1} = \frac{2-1}{1+1} = \frac{1}{2}; \bullet \lim_{u \rightarrow -3} \sqrt{9-u^2} = \sqrt{9-(-3)^2} = \sqrt{0} = 0$$

★ ★ ② In the quotient form  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ . If both  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$ , then we must cancel out the "zero terms" in  $f(x)$  and  $g(x)$  by the following factoring technique.

eg. 3. Compute  $\lim_{x \rightarrow 2} \frac{x^2+x-6}{x^2-2x}$ . Remark: If we plug in  $x=2$ , we get  $\frac{2^2+2-6}{2^2-2 \cdot 2} = \frac{0}{0}$ , which

solution: Factorize  $\lim_{x \rightarrow 2} \frac{(x-2) \cdot (x+3)}{x \cdot (x-2)}$  ~~cancel out~~  $\lim_{x \rightarrow 2} \frac{x+3}{x}$  Plug in  $\frac{2+3}{2} = \boxed{\frac{5}{2}}$  is meaningless.

eg. 4. If  $f(x) = \frac{1}{x+3}$ , then  $\lim_{x \rightarrow 1} \frac{f(x)-f(1)}{x-1} = \lim_{x \rightarrow 1} \frac{\frac{1}{x+3} - \frac{1}{4}}{x-1} \xrightarrow{\text{plug in}} \frac{-1}{4 \cdot 4} = \boxed{-\frac{1}{16}}$

solution:  $\frac{f(x)-f(1)}{x-1} = \frac{\frac{1}{x+3} - \frac{1}{4}}{x-1} = \frac{\frac{4-x-3}{4(x+3)}}{x-1} = \frac{1-x}{4(x+3)(x-1)} = \frac{-1}{4(x+3)}$  Hint:  $\frac{a/b}{c} = \frac{a}{b \cdot c}$

eg. 5. Compute  $\lim_{x \rightarrow -3^+} \frac{x-2}{x^2 \cdot (x+3)} = -\infty$ . Hint:  $x-2 = (-3)-2 = -5$ .

$$\left( \frac{-5}{9 \cdot (\text{small positive})} = -\infty \right)$$

$$x^2 = (-3)^2 = +9$$

$$x+3 = (-3)+3 = 0$$

while  $x+3$  small but positive since  $x \rightarrow -3^+$

★ eg. 6. For what value of  $C$  does  $\lim_{x \rightarrow 2} \frac{Cx^2+4}{x-2}$ , exist and is finite? (F/B) (approaches  $-3$  from right)

solution: Notice if we plug in  $x=2$ , we have  $\frac{4C+4}{2-2}$ , where the denominator is 0.

So we need the numerator also be zero, i.e.,

$$4C+4=0 \Rightarrow \boxed{C=-1}$$

Actually, if  $C=-1$ ,  $\lim_{x \rightarrow 2} \frac{(-1) \cdot x^2 + 4}{x-2} = \lim_{x \rightarrow 2} \frac{(2+x)(2-x)}{x-2}$

Hint:  $a^2 - b^2 = (a+b)(a-b)$

$$= \lim_{x \rightarrow 2} \frac{-(2+x) \cdot x}{x}$$

$$2^2 - x^2 = 4 - x^2 = (2+x)(2-x)$$

Therefore,  $\boxed{C=-1}$

$$= -4.$$

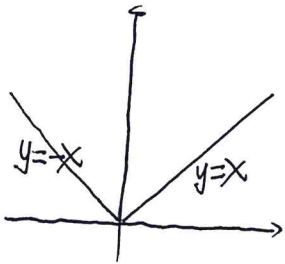
③ Squeeze Theorem: Suppose  $f(x) \leq g(x) \leq h(x)$  and  $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$   
Then  $\lim_{x \rightarrow a} g(x) = L$

eg7. If ~~cos(x)~~  $\cos(x) \leq g(x) \leq 1-x^2$ , find  $\lim_{x \rightarrow 0} g(x)$ .

$$\lim_{x \rightarrow 0} \cos x = \lim_{x \rightarrow 0} 1-x^2 = 1 \Rightarrow \lim_{x \rightarrow 0} g(x) = 1.$$

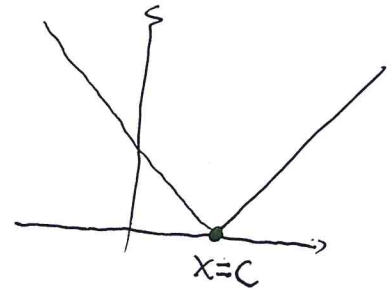
④ Absolute Value Function:  $f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

graph:



shift:  $y = |x - c|$  for some constant  $c$ .

$$= \begin{cases} x - c & x \geq c \\ -(x - c) & x < c \end{cases}$$



eg8. Find the limit of  $\lim_{x \rightarrow 1^+} \frac{2x \cdot (x-1)}{|x-1|}$

$x \rightarrow 1^+$  means  $x$  approaches 1 from the RIGHT:  $x > 1$

$$|x-1| = x-1$$

$$\Rightarrow \lim_{x \rightarrow 1^+} \frac{2x \cdot (x-1)}{|x-1|} = \lim_{x \rightarrow 1^+} \frac{2x \cdot \cancel{(x-1)}}{\cancel{x-1}} = \lim_{x \rightarrow 1^+} 2x = 2 \cdot 1 = 2$$

## § 1.8 A, B. Continuity.

Key points: ① Definition and the Graph.

② Continuity of piecewise function

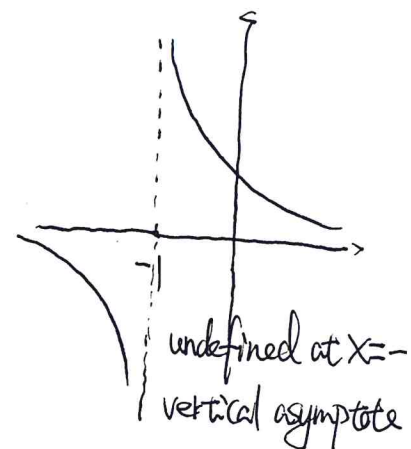
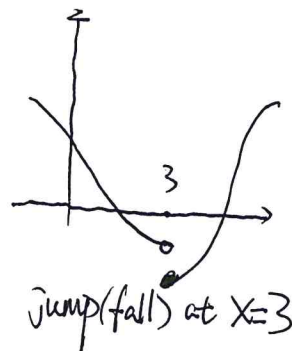
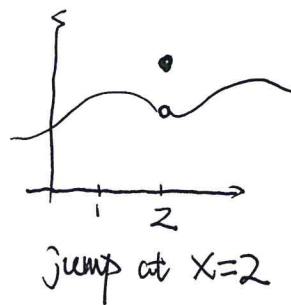
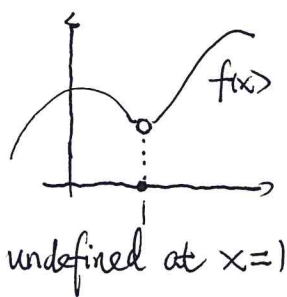
③ Intermediate Value Theorem.

• Definition:  $f(x)$  is continuous at  $x=a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$

In the graph, it means  $y=f(x)$  (the curve) does not have a jump/hole at  $x=a$

If it has a jump/hole,  $f(x)$  is discontinuous at  $x=a$ .

eg1. (Examples of discontinuity). The following functions are discontinuous at  $x=a$



• Domain of continuity. Function defined by formulas are continuous except at those undefined points

eg2. Functions:  $y=x$      $y=x^2$      $y=\sqrt{x}$      $y=\frac{1}{x}$      $y=\sin x$

D.O.C. :  $(-\infty, +\infty)$      $(-\infty, \infty)$      $[0, \infty)$      $(-\infty, 0) \cup (0, \infty)$      $(-\infty, \infty)$

$$g(x) = \frac{(x^2 - 3x + 1) \cdot \sqrt{x+1}}{x-3}$$

Domain:  $\sqrt{x+1} \Rightarrow x+1 \geq 0$ ,  $x-3 \neq 0$

D.O.C. :  $[-1, 3) \cup (3, +\infty)$

eg3. let  $g(x) = \begin{cases} x^3 + 2x & \text{if } x \leq 5 \\ \frac{5x^2 - x^3}{x-5} & \text{if } x > 5 \end{cases}$ . Is  $g(x)$  continuous at  $x=5$  or not? why?

Remark:  $g(x)$  is continuous at  $x=5$  if  $\lim_{x \rightarrow 5} g(x) = g(5)$ .

We need to study  $\lim_{x \rightarrow 5} g(x)$  first.

Solution:  $\lim_{x \rightarrow 5^-} g(x) = \lim_{x \rightarrow 5^-} x^3 + 2x = 5^3 + 2 \cdot 5 = 135$ .

$$\begin{aligned} \lim_{x \rightarrow 5^+} g(x) &= \lim_{x \rightarrow 5^+} \frac{5x^2 - x^3}{x-5} = \lim_{x \rightarrow 5^+} \frac{x^2(5-x)}{x-5} \quad \text{cancel the "zeros"} \\ &= \lim_{x \rightarrow 5^+} -x^2 = -25. \end{aligned}$$

$\lim_{x \rightarrow 5^-} g(x) \neq \lim_{x \rightarrow 5^+} g(x)$ , therefore, the limit  $\lim_{x \rightarrow 5} g(x)$  D.N.E.

$g(x)$  is NOT continuous at  $x=5$ .

eg4. For what values of  $k$  will  $f(x) = \begin{cases} \frac{x^2 - 3k}{x-3} & \text{if } x \leq 2 \\ 8x - k & \text{if } x > 2 \end{cases}$  be continuous for all  $x$ ?

Solution:  $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{x^2 - 3k}{x-3} = \frac{4-3k}{2-3} = \frac{4-3k}{-1} = 3k-4$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} 8x - k = 16 - k$$

We need the limit at  $x=2$  exists, i.e.,  $3k-4 = 16-k$ .

Solve for  $k$ :  $4k = 20$

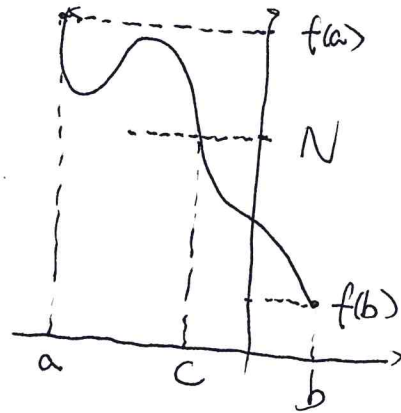
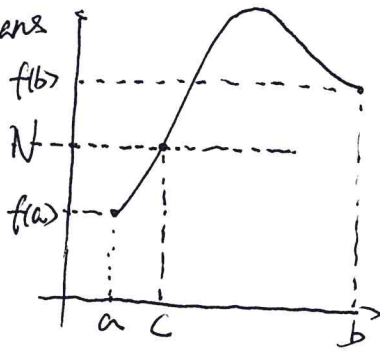
$$\Rightarrow \boxed{k=5}$$

## §1.8 B. ③ Intermediate Value Theorem.

(something in-between)

• (IVT) If  $f$  is continuous on  $[a, b]$ ,  $f(a) \neq f(b)$ , and  $N$  is between  $f(a)$  and  $f(b)$ , then there is a  $c \in (a, b)$  that satisfies  $f(c) = N$ .

• Graph: IVT means



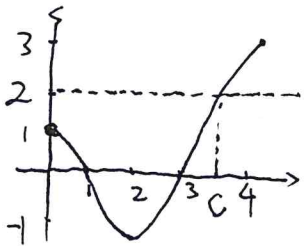
• Rewrite IVT as the following corollary:

In order to solve  $f(x) = N$ , it is enough to pick  $a, b$  such that  $f(a), f(b)$ , one larger and one smaller than  $N$ . Then there is a solution  $c$  in  $(a, b)$  to  $f(x) = N$ .

eg1. Let  $f(x)$  be continuous with values given below:

|        |   |   |    |   |   |
|--------|---|---|----|---|---|
| $x$    | 0 | 1 | 2  | 3 | 4 |
| $f(x)$ | 1 | 0 | -1 | 0 | 3 |

Sketch the graph of  $y = f(x)$  and find where is  $c$  st.  $f(c) = 2$ .



there is a  $c \in (3, 4)$ , such that  $f(c) = 2$ .

eg2. On which interval must there be a solution to  $x^3 - 14 = 36 - 3x$ ?

Consider  $f(x) = x^3 - 14 - (36 - 3x)$

$$f(3) = 3^3 - 14 - (36 - 3 \cdot 3) = 27 - 14 - (36 - 9) = \cancel{27} - 14 < 0$$

$$f(4) = 4^3 - 14 - (36 - 3 \cdot 4) = 26 > 0$$

therefore, there is a  $c \in (3, 4)$  such that  $f(c) = 0 \Leftrightarrow c^3 - 14 = 36 - 3c$