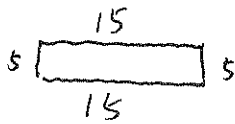


### §3.7. Optimization.

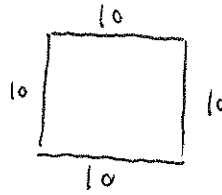
• Goal: Find abs max/min in real life

• Motivation: Fence problem: Suppose we have 40 meters of fence to make a rectangular corral. What length and width will fence off the largest area?

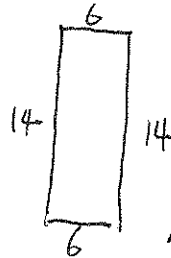
Possible figures:



Area: 75



Area: 100



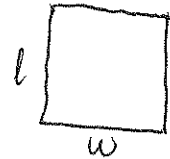
Area: 84

Math model: Give a rectangle with length  $l$  and width  $w$ .

Suppose the perimeter is constant, i.e.,  $2l + 2w = 40$ .

What are the values of  $l$  and  $w$  that make

the area  $A = l \cdot w$  reach its maximum?



Perimeter:  $2l + 2w$   
Area:  $l \cdot w$

Optimization problem: Give the constraint  $2l + 2w = 40$ . Find the  $l$  and  $w$  to maximize the function  $A = l \cdot w$ .

Key step: Use constraint to convert  $A$  into a one variable function.

$$2l + 2w = 40 \Rightarrow l + w = 20 \Rightarrow w = 20 - l \Rightarrow A = l(20 - l)$$

Variable:  $l$ . Function:  $A = l(20 - l) = 20l - l^2$ . Domain:  $[0, 20]$

Now use the method in §3.1 to find the abs max of  $A$  over  $[0, 20]$ .

Critical points:  $A' = (20l - l^2)' = 20 - 2l = 0 \Rightarrow l = 10$

endpoints:  $l = 0, l = 20$

List the values of  $A$  at the above points:  $A(10) = 20 \cdot 10 - 10^2 = 100$

$$A(0) = 0, A(20) = 0$$

i.e. The abs max of  $A$  is 100, it is attained at  $l = 10$ .

i.e. The area is maximized at width dimension  $l = 10, w = 10$

The maximum area is 100.

Method for optimization:

Step 1: Draw the picture with all varying quantities. Find the target quantity (target function) to be maximized or minimized.

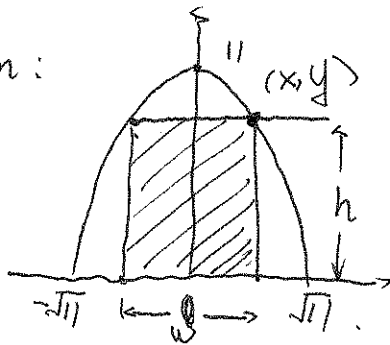
Step 2: Find the constraint (the relations/equations relating variables). Choose one appropriate quantity (variable) and use the constraint to express all the other quantities in terms of this main quantity (controlling variable). Express the target function only using this main variable and find its domain.

Step 3: Find the abs max/min via method in §3.1 (critical + endpoints). Draw the conclusion when the function is maximized/minimized.

eg. 1. A rectangle is inscribed with its base on  $x$ -axis and its upper corners on the parabola  $y=11-x^2$ . What are the dimensions of such a rectangle with the greatest possible area?

WARNING: The problem is not asking for the max of  $y=11-x^2$ .

Solution:



Draw the curve  $y=11-x^2$  first.  $y$  intercept:  $y=11$   
 $x$  intercepts:  $x=\pm\sqrt{11}$   
 $-x^2$  (negative coefficient)  $\Rightarrow$  opens down,  
 even function, symmetric about  $y$ -axis.

Dimension of the inscribed rectangle: width  $w$ , height  $h$ , Area  $A=w \cdot h$

Constraint: Upper corner on the parabola:  $w, h$  should be related to  $y=11-x^2$ .

Right upper corner with coordinate:  $(x, y) \Rightarrow h=y, w=2x$ .

$$A(x) = 2x \cdot (11 - x^2). \text{ Domain: } 0 \leq x \leq \sqrt{11}. \quad h=11-x^2,$$

$$= 22x - 2x^3.$$

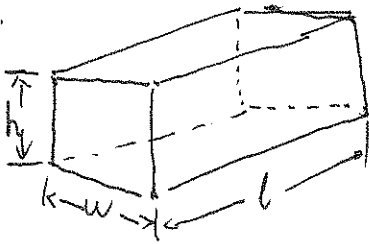
$$(\S 3.1) \quad A'(x) = 22 - 6x^2 = 0 \Rightarrow x^2 = \frac{22}{6} = \frac{11}{3} \Rightarrow x = \sqrt{\frac{11}{3}} \quad (\text{negative solution is discarded})$$

$$A\left(\sqrt{\frac{11}{3}}\right) = 2\sqrt{\frac{11}{3}} \left(11 - \frac{11}{3}\right) = \frac{44}{3} \sqrt{\frac{11}{3}} \quad (\text{abs max}) \text{ at } x = \sqrt{\frac{11}{3}}.$$

$$\text{Dimension: } w = 2x = 2\sqrt{\frac{11}{3}}, \quad h = 11 - x^2 = 11 - \frac{11}{3} = \frac{22}{3}$$

e.g.2. A total of  $1200 \text{ cm}^2$  of material is to be used to make a box  
(Fall 15). with no top. Assume that the base is to be twice as long as its wide.  
Find the largest volume of such a box.

Picture:



Dimension: height  $h$ , width  $w$ , length  $l$

Volume (Target function):  $V = h \cdot w \cdot l$  ... (1)

Surface area (without top) is a constraint:

$$2h \cdot w + 2h \cdot l + w \cdot l = 1200 \quad \dots (2)$$

Relation between length and width:  $l = 2w$  ... (3)

Goal: choose  $w$  as the main variable and express  $l, h, V$  by  $w$ .

$$l = 2w \Rightarrow 2h \cdot w + 2h \cdot 2w + w \cdot 2w = 1200 \quad (\text{Plug } l=2w \text{ into (2)})$$

$$\Rightarrow 6h \cdot w + 2w^2 = 1200 \quad (\text{Fix } w, \text{ solve for } h)$$

$$\Rightarrow h = \frac{1200 - 2w^2}{6w}$$

Plug  $l=2w, h = \frac{1200 - 2w^2}{6w}$  into  $V = h \cdot w \cdot l$

$$V(w) = \frac{1200 - 2w^2}{6w} \cdot w \cdot 2w \quad \text{simplify} \quad \frac{(1200 - 2w^2)}{3} \cdot w = \frac{1200}{3}w - \frac{2}{3}w^3 = \boxed{400w - \frac{2}{3}w^3}$$

Now use the method in §3.1 to find  $V(w)$ 's abs. max and the corresponding  $w$ .

★ Notice all  $w, l, h$  have to be positive,  $h = \frac{1200 - 2w^2}{6w} > 0 \Rightarrow 1200 - 2w^2 < 0 \Rightarrow w^2 < 600$

Domain of  $V(w)$ :  $w \in [0, \sqrt{600}]$ ,  $\Rightarrow w < \sqrt{600} = \sqrt{6} \cdot 10$

Critical points of  $V$ :  $V' = (400w - \frac{2}{3}w^3)' = 400 - \frac{2}{3} \cdot 3w^2 = 0$

$$\Rightarrow 400 - 2w^2 = 0 \Rightarrow w^2 = 200 \Rightarrow w = \sqrt{200} = 10\sqrt{2} \in [0, \sqrt{600}]$$

Endpoints:  $w=0, w = \sqrt{600} \Rightarrow V(0) = 0, V(\sqrt{600}) = 400 \cdot \sqrt{600} - \frac{2}{3}(\sqrt{600})^3 = 0$

$$V(\sqrt{200}) = 400 \cdot \sqrt{200} - \frac{2}{3}(\sqrt{200})^3 = 400 \cdot \sqrt{200} - \frac{2}{3}(\sqrt{200})^2 \cdot \sqrt{200}$$

$$\text{The abs max} = \frac{800}{3}\sqrt{200}, \quad = 400 \cdot \sqrt{200} - \frac{2}{3} \cdot 200 \cdot \sqrt{200} = \boxed{\frac{800}{3}\sqrt{200}} = \frac{80000}{3}\sqrt{2}$$

attained at  $w = \sqrt{200} \Rightarrow l = 2 \cdot \sqrt{200}, h = \frac{1200 - 2 \cdot (\sqrt{200})^2}{6 \cdot \sqrt{200}} = \frac{800}{6 \cdot \sqrt{200}}$

## § Newton's Method

• Formula: Newton's Method:  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$  ----- (1)

• The above formula is a numerical method for finding successively better approximations to the roots (or zeros) of a function  $f(x)$ , i.e., the solutions  $x$  to the equation  $f(x) = 0$ .

★ Give  $f(x)$ ,  $f'(x)$  and INITIAL value  $x_1$ , the formula (1) runs inductively for  $n=1, 2, 3, \dots$

$$n=1, x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}; \quad n=2, x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}; \quad \dots \quad (2)$$

Conclusion: The list of numbers given by (2),  $x_1, x_2, x_3, \dots$  are getting closer and closer (approaching) the solutions of  $f(x) = 0$ .

★ Key point: Know how to use (2) to compute  $x_2, x_3$ .

eg. 1 Suppose you are using Newton's method to estimate the value of  $\sqrt{2}$  (s16) by finding the root of  $x^2 - 2 = 0$ . If you use  $x_1 = 1$ , find  $x_2$ .

Solution: Preparation:  $f(x) = x^2 - 2$ ,  $f'(x) = 2x$ ,  $x_1 = 1$ . Plug into (2).

$$(2) \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1-2}{2 \cdot 1} = 1 - \frac{-1}{2} = \boxed{\frac{3}{2}}$$

Remark: If the problem asks for  $x_3$ , then plug  $x_2 = \frac{3}{2}$  into (2) one more time.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = \frac{3}{2} - \frac{f(\frac{3}{2})}{f'(\frac{3}{2})} = \frac{3}{2} - \frac{(\frac{3}{2})^2 - 2}{2 \cdot \frac{3}{2}} = \frac{3}{2} - \frac{\frac{9}{4} - 2}{3} = \frac{3}{2} - \frac{\frac{1}{4}}{3} = \frac{17}{12}$$

eg. 2. Use Newton's method to approximate a nonzero solution of the equation  $4\sin x = x$ . Let  $x_1 = 2$ . Find  $x_2$  to four decimal places (use calculator).

Key step: Rewrite  $4\sin x = x$  as  $f(x) = 0$ .  $4\sin x - x = 0$ , i.e.,  $f(x) = 4\sin x - x$ .

Solution:  $f'(x) = 4\cos x - 1$ ,  $x_1 = 2 \Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{4\sin 2 - 2}{4\cos 2 - 1} \approx 2.6144$ .