Name: ______________________________ **ID:** ________________

Clear your desk of everything except pens, pencils and erasers. Show all work clearly and in order. No notes, phones and calculators. You have 10 minutes to finish these two problems for 10 points.

1. Consider the power series $\sum_{n=1}^{\infty} \frac{(5x-1)^n}{n^2}$

   (a) (3 points) Test the above series at $x = \frac{2}{5}$ for convergence or divergence.

$$\sum_{n=1}^{\infty} \frac{(5x-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(5\cdot\frac{2}{5}-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{1^n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$ p-Series, $p=2>1$

convergent.

   (b) (4 points) Find the OPEN interval of the convergence for the above series.

$$a_n = \frac{(5x-1)^n}{n^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(5x-1)^{n+1}}{(n+1)^2}}{\frac{(5x-1)^n}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}(5x-1)\right|$$

$$= |5x-1| < 1$$

$$\Rightarrow \quad -1 < 5x-1 < 1$$

$$\Rightarrow \quad 0 < 5x < 2 \quad \Rightarrow \quad 0 < x < \frac{2}{5}$$

open interval of conv: $(0, \frac{2}{5})$.

2. (3 points) Give $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. Integrate $\frac{1}{1+x^2}$ as a power series. Which of the following is the expression for the first THREE NON-ZERO terms in the power series of $\int \frac{1}{1+x^2}$.

   A. $1 - x^2 + x^4 \cdots$

   B. $1 + x^2 + x^4 \cdots$

   ✓ C. $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \cdots$

   D. $1 - x^2 + x^3 \cdots$

   E. $x - \frac{1}{3}x^3 + \frac{1}{4}x^4 \cdots$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\int \frac{1}{1+x^2}dx = \int 1 - x^2 + x^4 - x^6 + \cdots \, dx$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

Series

- **nth term test for divergence:** If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The p-series:** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

- **Geometric:** If $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

- **The Integral Test:** Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then

  (i) If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

  (ii) If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

- **The Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

  (i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

  (ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

- **The Limit Comparison Test:** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both series converge or both diverge.

- **Alternating Series Test:** If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ satisfies

  (i) $0 < b_{n+1} \leq b_n$ for all n

  (ii) $\lim_{n\to\infty} b_n = 0$

  then the series is convergent.

- **The Ratio Test**

  (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

  (ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

  (iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

- **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n$

- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

- Some Power Series
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\qquad R = \infty$
  - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\qquad R = \infty$
  - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\qquad R = \infty$
  - $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ $\qquad R = 1$
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\qquad R = 1$