Sec8.1. Arc-length. LecNote6.

Q1 Find the exact arc-length of $x = \frac{2}{3}(y^2+1)^{3/2}$ from $y = 0$ to $y = 2$.

Formula: $\text{Arc-length} = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \cdot dy$

Algebra Hints: complete the square:

$$1 + 4y^2 + 4y = (2y)^2 + 2\cdot 2y + 1^2$$

$$= (2y+1)^2$$

Q2(Limit of a sequence.) Find the limit if the sequence below converges or state why it diverges.

(a)

$$a_n = \frac{1}{n}\ln(\frac{1}{n})$$

Hint: $a_n = \frac{\ln(n^{-1})}{n} = \frac{-\ln n}{n}$ $\frac{\infty}{\infty}$ l'Hôpital.

(b)

$$a_k = \frac{\sqrt{1+k^3}}{3k^2+7k}$$

Hint: leading terms rule.

or rewrite $a_k = \sqrt{\frac{1+k^3}{(3k^2+7k)^2}}$ $\frac{\infty}{\infty}$ l'Hôpital

(c)

$$a_n = n(e^{\frac{1}{n}} - 1)$$

$\infty(e^0 - 1) = \infty \cdot 0$

Hint: $a_n = \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}}$ $\frac{0}{0}$ l'Hôpital

Sec11.2. (Introduction to) Series. LecNote7.

Q3 (n-th term test for divergence). Which statements (more than one option) are true about

$$a_n = e^{\frac{2}{n}}, \quad \text{and} \quad \sum_{n=1}^{\infty} e^{\frac{2}{n}}$$

$\lim a_n = e^0 = 1 \neq 0$

A. The sequence a_n is convergent but the series is divergent.

B. The sequence a_n converges, therefore, the **nth term test** concludes that the series converges.

C. The sequence a_n has limit zero, therefore, the **nth term test** concludes that the series converges.

D. The **nth term test** concludes that the series diverges.

E. If the series is convergent, then the sequence a_n should have zero limit, which is a contradiction. Therefore, the series can not be convergent.

Q4 (Geometric Sequence/Sum/Series). Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{9^{n/2}}{3(2^{2n+1})}$$

Hint: $\dfrac{9^{\frac{n}{2}}}{3\cdot 2^{2n+1}} = \dfrac{(\sqrt{9})^n}{3\cdot 2\cdot 2^{2n}} = \dfrac{3^n}{6\cdot(2^2)^n}$

$= \frac{1}{6}\cdot\left(\frac{3}{4}\right)^n$

$= \underbrace{\frac{1}{6}\left(\frac{3}{4}\right)}_{a} \cdot \underbrace{\left(\frac{3}{4}\right)^{n-1}}_{r}$

or. $\displaystyle\sum_{n=1}^{\infty} \frac{9^{\frac{n}{2}}}{3\cdot 2^{2n+1}} = \underbrace{\frac{9^{\frac{1}{2}}}{3\cdot 2^3}}_{\substack{n=1 \\ a}} + \underbrace{\frac{9^{\frac{2}{2}}}{3\cdot 2^5}}_{\substack{n=2 \\ a\cdot r}} + \cdots$

Sec11.3. Integral Test and the p-Series. LecNote7.

Q5 (Integral Test) Test the following series for convergence or divergence by THE INTEGRAL TEST.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{1}{n}\right)$$

$f(x) = \frac{1}{x^2} \sin\left(\frac{1}{x}\right)$ is positive continuous, decreasing

Test $\int_1^{\infty} \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx$. (in Mid-1)?

Q6 (p-series) Which statements (more than one option) are true

A. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $|p| < 1$.

B. The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if and only if $|p| > 1$.

$\}$ p-Series $\sum \frac{1}{n^p}$

C. The series $\sum_{n=1}^{\infty} r^n$ is convergent if $|r| < 1$.

D. The series $\sum_{n=0}^{\infty} r^n$ is divergent at $r = -1$.

$\}$. Geometric Series $\sum a \cdot r^n$

E. The series $\sum_{n=1}^{\infty} \frac{-3\sqrt{n}}{n^{1.5}}$ is convergent since it is a constant multiple of a p-series with $p = 1.5 > 1$.

F. The p-series $\sum_{n=1}^{\infty} n^{-2}$ is divergent since $p = -2$ and $|p| = 2 > 1$.

Q7 Determine whether the following series converge or diverge by (Direct/Limit) Comparison Test.

(a)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n}$$

Direct : $\frac{\sqrt{n+1}}{n} > \frac{1}{n}$.

Limit : $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{3n^3 - 7n}$$

Limit C.T.. $b_n = \frac{\sqrt{n^2}}{3n^3} = \frac{n}{3n^3} = \frac{1}{3n^2}$

(c)

$$\sum_{n=2}^{\infty} \frac{2}{n^{61}+1}$$

Direct. $\frac{2}{n^{61}+1} < \frac{2}{n^{61}}$

Limit: $b_n = \frac{2}{n^{61}}$

Sec11.5. Alternating Series Test and Absolute Convergence. LecNote8.

Q8 Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

(a)

$$(1)\ \sum_{n=1}^{\infty} \frac{\cos(5n)}{n^5} \qquad \text{and (2)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n}$$

A (1) is absolutely convergent; (2) is divergent.

B (1) is conditionally convergent; (2) is divergent.

C (1) is absolutely convergent; (2) is conditionally convergent.

D (1) is divergent; (2) is conditionally convergent.

E (1) and (2) are conditionally convergent.

Hint: Test $\sum \left| \frac{\cos(5n)}{n^5} \right|$. ABS conv $\Rightarrow$ conv.

- $\sum a_n$ is ABS conv if $\sum |a_n|$ is conv.
- Alternating Test for (2). $\sum (-1)^n \cdot b_n$, b_n decreasing and $\lim b_n = 0$

(b) Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

$$(1)\ \sum_{n=1}^{\infty} \frac{\sin(n)+1}{2^n} \qquad \text{and (2)} \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

A (1) is absolutely convergent; (2) is divergent.

B (1) is conditionally convergent; (2) is divergent.

C (1) is absolutely convergent; (2) is conditionally convergent.

D (1) is divergent; (2) is conditionally convergent.

E (1) and (2) are conditionally convergent.

Hint: $0 \le \sin n + 1 \le 2$

Sec11.6. Ratio Test. LecNote8.

Q9 Determine whether the following series converge or diverge.

(a)

$$\sum_{n=1}^{\infty} \frac{2^n(n^2+1)}{3^n}$$

$\lim \frac{a_{n+1}}{a_n} = L$ compare with 1.

(b)

$$\sum_{n=1}^{\infty} \frac{(n+1)!}{e^n}$$

(c)

$$\sum_{n=1}^{\infty} \frac{n}{(-2)^n}$$

$|(-2)^n| = 2^n$

$\lim \left| \frac{a_{n+1}}{a_n} \right|$ compare with 1.

Sec11.8. Power Series. LecNote9.

Q10 Consider the following power series

$$\sum_{n=0}^{\infty}(n+3)\left(\frac{2x-3}{3}\right)^n$$

(a) Find the radius of convergence of the series and the OPEN interval of the convergence.

$$a_n = (n+3)\cdot\left(\frac{2x-3}{3}\right)^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

(b) Test the Left and Right Endpoints of the open interval in Part (a) for convergence or divergence.

x = left endpoint

x = right endpoint.

Sec11.9. Power Series Representation. *LecNote9.*

Q11 Consider the function $f(x) = \frac{x}{1+x}$

(a) Find the first FOUR non-zero terms of the power series representation of the function $f(x)$

$$\frac{1}{1-\blacksquare} = \sum_{n=0}^{\infty} \blacksquare^{n} \Rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

(b) Use the expression in Part (a) to find the first THREE non-zero terms of the power series representation of the DERIVATIVE function of f,

$$f'(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$\left(\sum \quad \right)' = \sum (\quad)'$$

(c) Use the expression in Part (a) to find the first THREE non-zero terms of the power series representation of the indefinite INTEGRAL of f,

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n x^n + C$$

$$\int \sum (\quad) dx = \sum \int (\quad) dx$$

Sec11.10/11. Taylor and Maclaurin Series. *LecNote10.* **Q12** Find the 3rd degree Taylor polynomial of $f(x) = x\cos(2x)$ centered at $x = \pi/6$

$n=3$, $a=\frac{\pi}{6}$

$T_3(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

Derivative Table for $f(a)$, $f'(a)$, $f'(a)$, $f'''(a)$

Q13 Consider the function $f(x) = \ln(1-2x)$.

(a) Find the Maclaurin series of $f(x)$.

Formula sheet: $\ln(1+\blacksquare) = \sum_{n=1}^{\infty} \frac{\blacksquare^n}{n}$

$\blacksquare = -2x$.

(b) 'Evaluate'(Guess) the limit $\lim_{x\to 0} \frac{\ln(1-2x)}{x}$ by the power series expression in Part (a).

(a) $\Rightarrow$ $\ln(1-2x) \approx -2x$.

(c) Verify the answer in (b) by l'Hopital's Rule.

$\frac{\ln(1-2x)}{x}$ $\quad \frac{\ln 1}{0} = \frac{0}{0}$

Q14 Consider the function $f(x) = xe^x$.

(a) Find the power series expansion of the function $f(x) = xe^x$ centered at $x = 0$.

(= Maclaurin)

$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, formula sheet.

(b) Find the 3rd degree Taylor polynomial $T_3(x)$ of $f(x)$ at $x = 0$. Find $T_3(0.5)$ for estimating $f(0.5)$.

$T_3(x) = C_0 + C_1 \cdot x + C_2 x^2 + C_3 \cdot x^3$

Plug in $x = 0.5$. $T_3(0.5)$.

(c) Suppose we know that $|f^4(x)| \leq 15$ for all $|x| \leq 1$, what the error in Part (b) when we use $T_3(0.5)$ to approximate $f(0.5)$? And what's the maximal error for estimating $f(x)$ via $T_3(x)$ on $[-1, 1]$.

Taylor's Ineq. $n = 3$.

$|R_3(x)| \leq \frac{M}{(3+1)!} \cdot |x-a|^{3+1}$, for $|x-a| \leq d$.

M: maximal of $f^{(n+1)}(x)$ on $|x-a| \leq d$

$|x| \leq 1$

$\Rightarrow a = 0, d = 1$